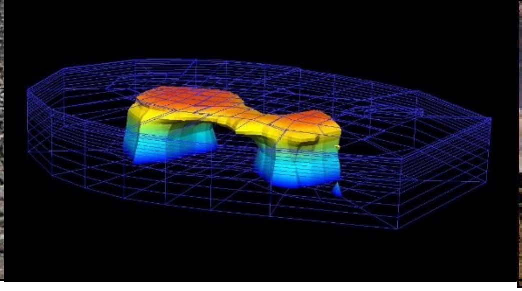
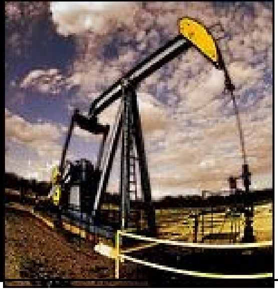
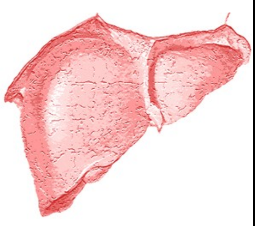
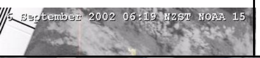




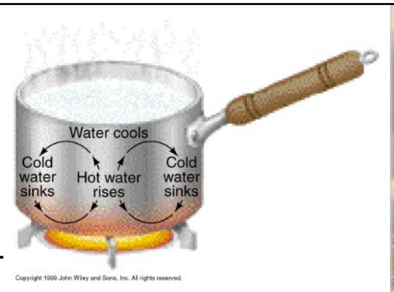


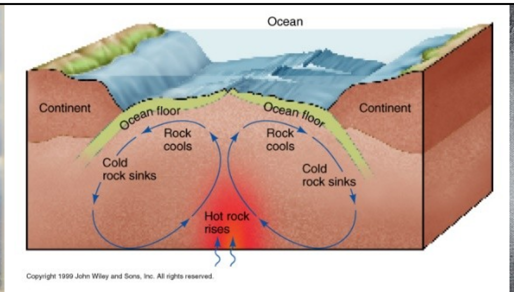


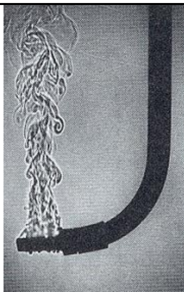



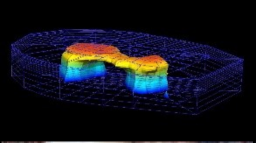
Introduction to Ordinary Differential Equations

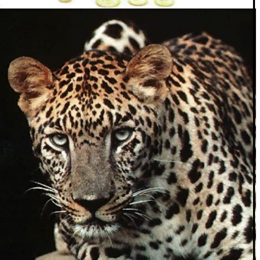


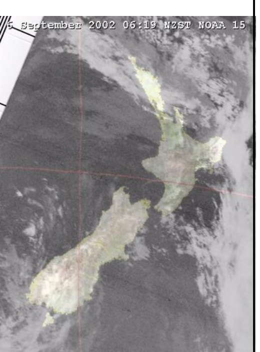




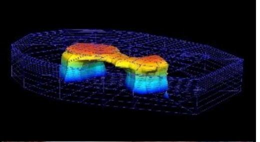
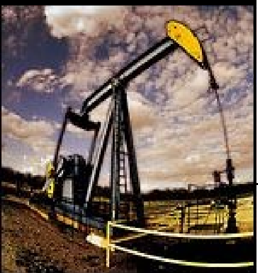


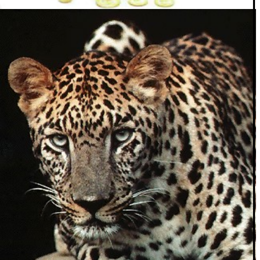
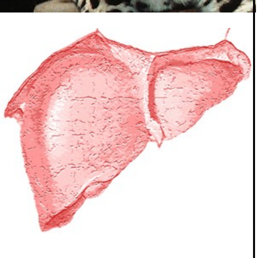
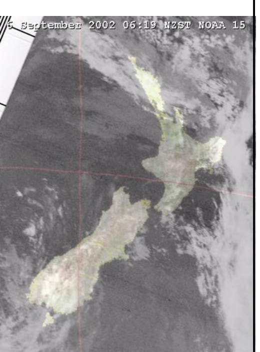





Ordinary Differential Equations

- Where do ODEs arise?
- Notation and Definitions
- Solution methods for 1st order ODEs

Where do ODE’s arise

- All branches of Engineering
- Economics
- Biology and Medicine
- Chemistry, Physics etc

Anytime you wish to find out how something changes with time (and sometimes space)

Example – Newton's Law of Cooli

- This is a model of how the temperature of an object changes as it loses heat to the surrounding atmosphere:

Temperature of the object: T_{Obj} Room Temperature: T_{Room}

<u>Newton's laws states</u>: "*The rate of change in the temperature of an object is proportional to the difference in temperature between the object and the room temperature*"

Form ODE

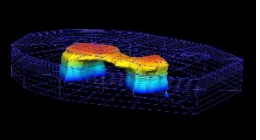
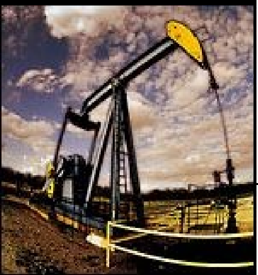
$$\frac{dT_{Obj}}{dt} = -\alpha(T_{Obj} - T_{Room})$$

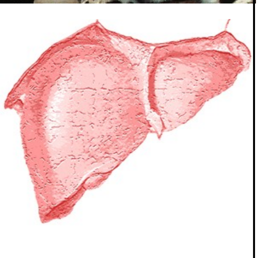
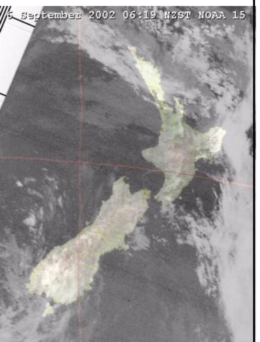
Solve ODE

$$T_{Obj} = T_{Room} + (T_{init} - T_{Room})e^{-\alpha t}$$

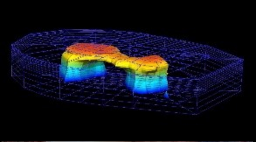
Where T_{init} is the initial temperature of the object.

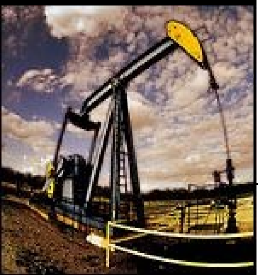
Notation and Definitions

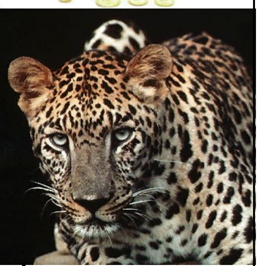



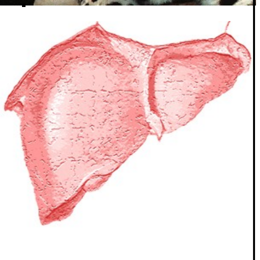




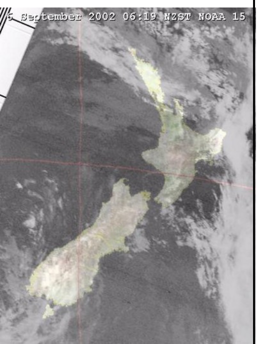
- Order
- Linearity
- Homogeneity
- Initial Value/Boundary value problems









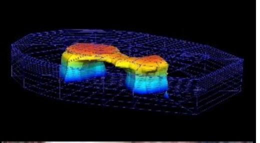


Order

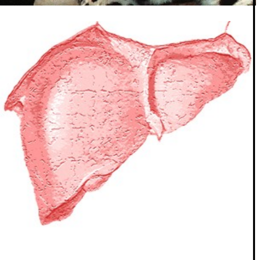
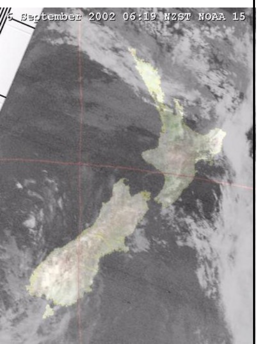
- The order of a differential equation is just the order of highest derivative used.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$ → 2nd order

$$\frac{dx}{dt} = x\frac{d^3x}{dt^3}$$ → 3rd order

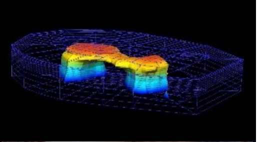
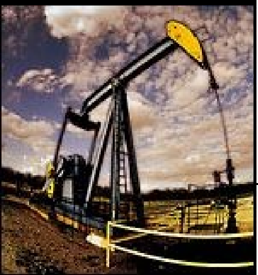


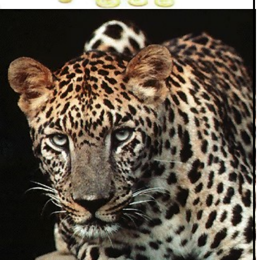
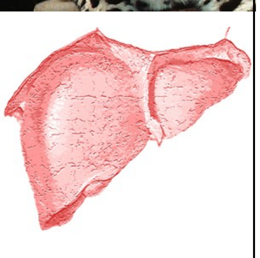
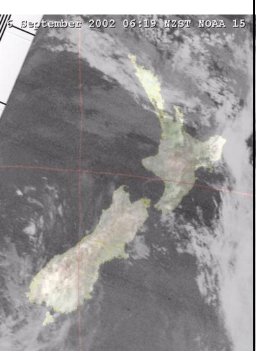


Linearity

- The important issue is how the unknown *y* appears in the equation. A linear equation involves the dependent variable (*y*) and its derivatives by themselves. There must be no "unusual" nonlinear functions of *y* or its derivatives.
- A linear equation must have constant coefficients, or coefficients which depend on the independent variable (*t*). If *y* or its derivatives appear in the coefficient the equation is non-linear.

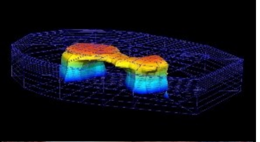
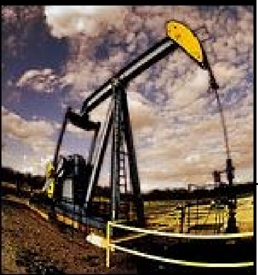
Linearity - Examples

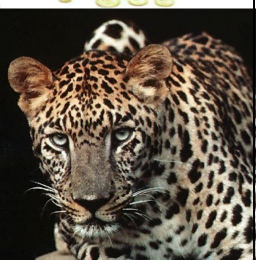
$$\frac{dy}{dt} + y = 0$$ is linear

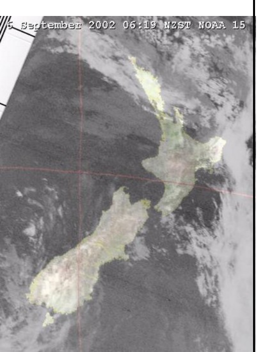
$$\frac{dx}{dt} + x^2 = 0$$ is non-linear

$$\frac{dy}{dt} + t^2 = 0$$ is linear

$$y\frac{dy}{dt} + t^2 = 0$$ is non-linear

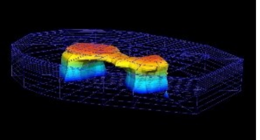



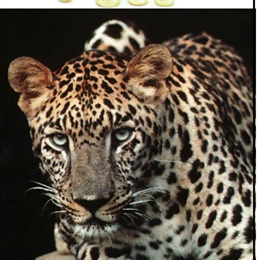
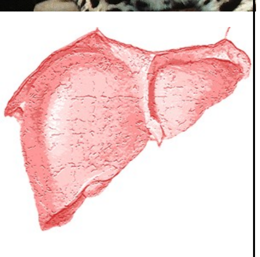
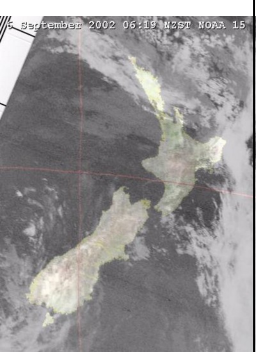




Linearity – Summary

Linear	Non-linear
$2y$	y^2 or $\sin(y)$
$\dfrac{dy}{dt}$	$y\dfrac{dy}{dt}$
$(2+3\sin t)y$	$(2-3y^2)y$
$t\dfrac{dy}{dt}$	$\left(\dfrac{dy}{dt}\right)^2$



Linearity – Special Property

If a linear homogeneous ODE has solutions:

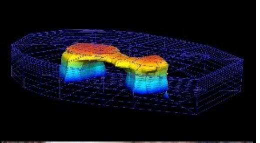
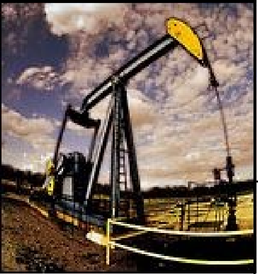
$$y = f(t) \quad \text{and} \quad y = g(t)$$

then:

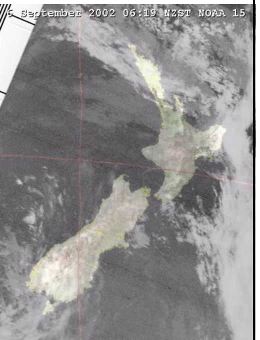
$$y = a \times f(t) + b \times g(t)$$

where a and b are constants,

is also a solution.





Linearity – Special Property

Example:

$$\frac{d^2y}{dt^2} + y = 0$$ has solutions $y = \sin t$ and $y = \cos t$

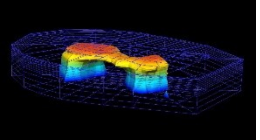
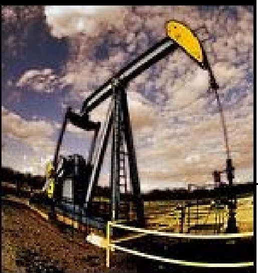
Check $$\frac{d^2(\sin t)}{dt^2} + \sin t = -\sin t + \sin t = 0$$

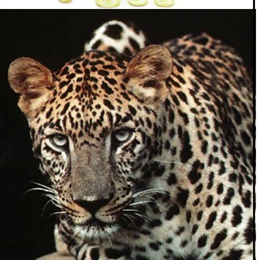
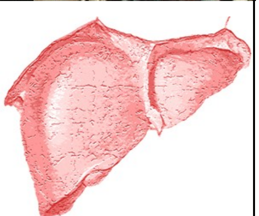
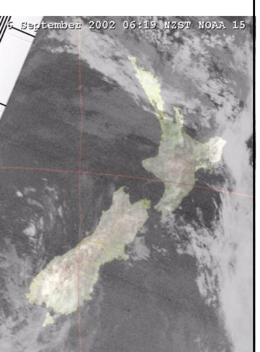
$$\frac{d^2(\cos t)}{dt^2} + \cos t = -\cos t + \cos t = 0$$

Therefore $y = \sin t + \cos t$ is also a solution:

Check $$\frac{d^2(\sin t + \cos t)}{dt^2} + \sin t + \cos t$$

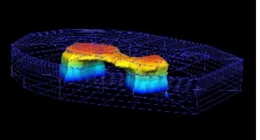
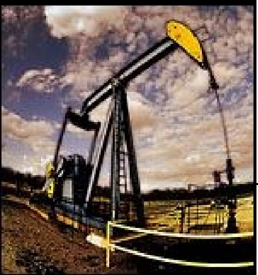
$$= -\sin t - \cos t + \sin t + \cos t = 0$$

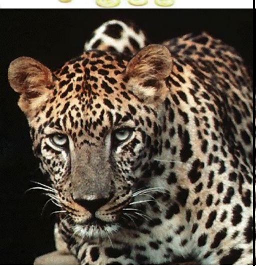
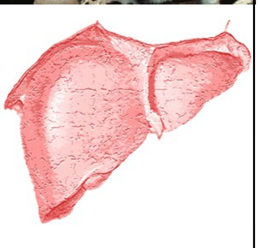
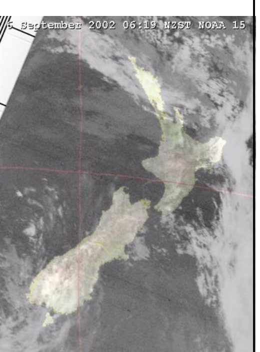



Homogeniety

- Put all the terms of the equation which involve the dependent variable on the LHS.
- **Homogeneous:** If there is nothing left on the RHS the equation is homogeneous (unforced or free)
- **Nonhomogeneous:** If there are terms involving t (or constants) - but not y - left on the RHS the equation is nonhomogeneous (forced)

Example

$$\frac{dv}{dt} = g$$

$$v(0) = v_0$$

- 1st order
- Linear
- Nonhomogeneous
- Initial value problem

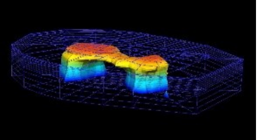
$$\frac{d^2M}{dx^2} = w$$

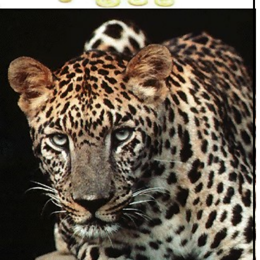
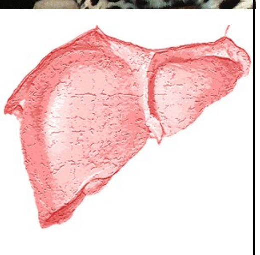
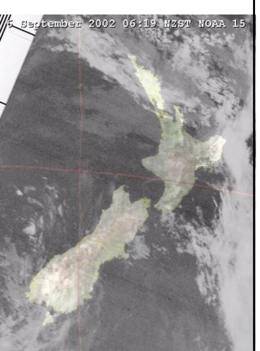
$$M(0) = 0$$

and

$$M(l) = 0$$

- 2nd order
- Linear
- Nonhomogeneous
- Boundary value problem



Example

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$

$$\theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0$$

- 2nd order
- Nonlinear
- Homogeneous
- Initial value problem

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$$

$$\theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0$$

- 2nd order
- Linear
- Homogeneous
- Initial value problem

Solution Methods - Direct Integration

- This method works for equations where the RHS does not depend on the unknown:
- The general form is:

$$\frac{dy}{dt} = f(t)$$

$$\frac{d^2y}{dt^2} = f(t)$$

$$\vdots$$

$$\frac{d^ny}{dt^n} = f(t)$$

Direct Integration

- *y* is called the unknown or dependent variable;
- *t* is called the independent variable;
- “solving” means finding a formula for *y* as a function of *t;*
- Mostly we use *t* for time as the independent variable but in some cases we use *x* for distance.

Direct Integration – Example

Find the velocity of a car that is accelerating from rest at 3 ms^{-2}:

$$\frac{dv}{dt} = a = 3$$
$$\Rightarrow v = 3t + c$$

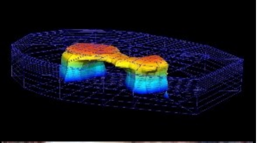
If the car was initially at rest we have the condition:

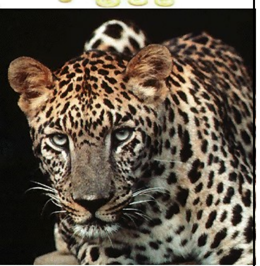
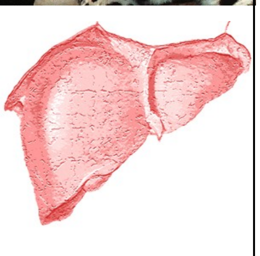
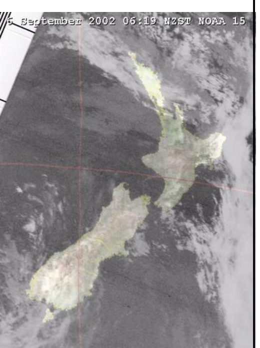
$$v(0) = 0 \Rightarrow 0 = 3 \times 0 + c \Rightarrow c = 0$$
$$\Rightarrow v = 3t$$

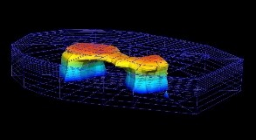
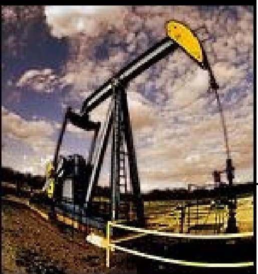
Solution Methods - Separation

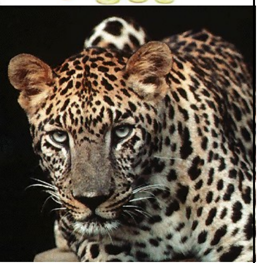
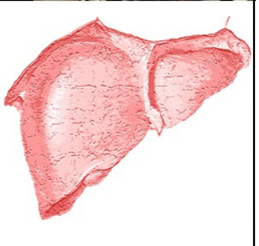
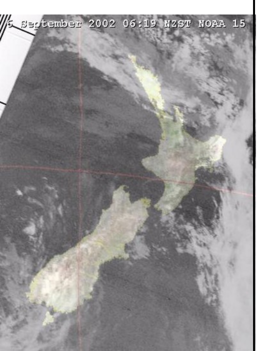
The separation method applies only to 1st order ODEs. It can be used if the RHS can be factored into a function of *t* multiplied by a function of *y:*

$$\frac{dy}{dt} = g(t)h(y)$$



Separation – General Idea

First Separate:

$$\frac{dy}{h(y)} = g(t)dt$$

Then integrate LHS with respect to y, RHS with respect to t.

$$\int \frac{dy}{h(y)} = \int g(t)dt + C$$

Separation - Example

$$\frac{dy}{dt} = y\sin(t)$$

Separate:

$$\frac{1}{y}dy = \sin(t)dt$$

Now integrate:

$$\int \frac{1}{y}dy = \int \sin(t)dt$$

$$\Rightarrow \ln(y) = -\cos(t) + c$$

$$\Rightarrow y = e^{-\cos(t)+c}$$

$$\Rightarrow y = Ae^{-\cos(t)}$$