2. Basic Group Theory

2.1 Basic Definitions and Simple Examples
2.2 Further Examples, Subgroups
2.3 The Rearrangement Lemma & the Symmetric Group
2.4 Classes and Invariant Subgroups
2.5 Cosets and Factor (Quotient) Groups
2.6 Homomorphisms
2.7 Direct Products

2.1 Basic Definitions and Simple Examples

Definition 2.1: Group

{ G, • } is a group if ∀ a , b , c Î G

1. a • b ÎG (closure)
2. (a • b) • c = a • (b • c) (associativity)
3. $ e Î G ' e • a = a • e = a (identity)
4. $ a^{-1} Î G ' a^{-1} • a = a • a^{-1} = e (inverse)

Definition in terms of multiplication table (abstract group):

G	e	a	b	⋯
e	e • e	e • a	e • b	⋯
a	a • e	a • a	a • b	⋯
b	b • e	b • a	b • b	⋯
⋮	⋮	⋮	⋮	⋱

e	a	b	⋯
a	a • a	a • b	⋯
b	b • a	b • b	⋯
⋮	⋮	⋮	⋱

Example 1: C_1

C_1	e
e	e

Realizations:

- {e} = { 1 }

Example 2: C_2

e	a
a	e

Realizations:

- {e,a} = { 1, –1}
- Reflection group: $C_\sigma = \{ E, \sigma \}$
- Rotation group: $C_2 = \{ E, C_2 \}$

C_n = Rotation of angle $2\pi/n$

Example 3: C_3

e	a	b
a	b	e
b	e	a

Realizations:

- Rotation group: $C_3 = \{ E, C_3 , C_3^{-1} \}$
- Cyclic group: $C_3 = \{ e, a, a^2 ; a^3=e \}$
- $\{ 1, e^{i 2\pi/3}, e^{i 4\pi/3} \}$
- Cyclic permutation of 3 objects { (123), (231), (312) }

Cyclic group : $C_n = \{ e, a, a^2, a^3, \dots a^{n-1}; a^n = e \}$

Definition 2.2: Abelian (commutative) Group

G is Abelian if a b = b a $\forall$ a,b $\in$ G

Common notations:

• → + e → 0

Definition 2.3: Order

Order g of group G = Number of elements in G

Example 4: Dihedral group D_2

Simplest non-cyclic group is

$D_2 = \{ e, a = a^{-1}, b = b^{-1}, c = a\,b \}$

(Abelian, order = 4)

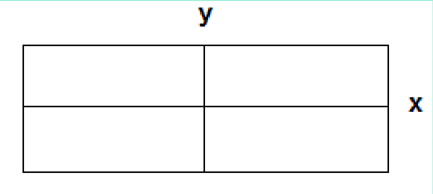
e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

Realizations:

D_2 = { symmetries of a rectangle }

$= \{ E, C_2, \sigma_x, \sigma_y \}$

$= \{ E, C_2, C_2', C_2'' \}$



2.2 Further Examples, Subgroups

The simplest non-Abelian group is of order 6.

$\{ e, a, b = a^{-1}, c = c^{-1}, d = d^{-1}, f = f^{-1} \}$

Aliases: Dihedral group D_3, C_{3v}, or permutation group S_3.

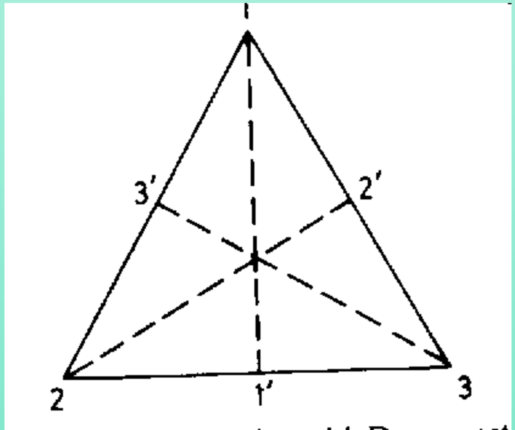
e	a	b	c	d	f
a	b	e	f	c	d
b	e	a	d	f	c
c	d	f	e	a	b
d	f	c	b	e	a
f	c	d	a	b	e

e	C_3	C_3^2	σ_1	σ_2	σ_3
C_3	C_3^2	e	σ_3	σ_1	σ_2
C_3^2	e	C_3	σ_2	σ_3	σ_1
σ_1	σ_2	σ_3	e	C_3	C_3^2
σ_2	σ_3	σ_1	C_3^2	e	C_3
σ_3	σ_1	σ_2	C_3	C_3^2	e

Symmetries of an equilateral triangle:

$C_{3v} = \{ E, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3 \}$

$D_3 = \{ E, C_3, C_3^2, C_2', C_2'', C_2''' \}$



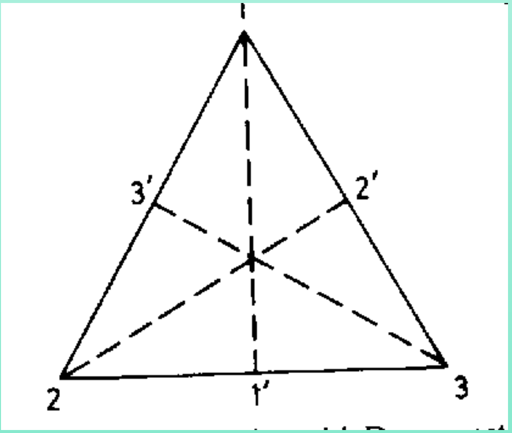
S_3 = { e, (123), (132), (23), (13), (12) }

(…) = cyclic permutations

e	(123)	(132)	(23)	(13)	(12)
(123)	(132)	e	(12)	(23)	(13)
(132)	e	(123)	(13)	(12)	(23)
(23)	(13)	(12)	e	(123)	(132)
(13)	(12)	(23)	(132)	e	(123)
(12)	(23)	(13)	(123)	(132)	e

e	(12)	(23)	(31)	(123)	(321)
(12)	e	(123)	(321)	(23)	(31)
(23)	(321)	e	(123)	(31)	(12)
(31)	(123)	(321)	e	(12)	(23)
(123)	(31)	(12)	(23)	(321)	e
(321)	(23)	(31)	(12)	e	(123)

Tung's notation



Definition 2.4: Subgroup

$\{ H \subseteq G, \bullet \}$ is a subgroup of $\{ G, \bullet \}$.

Example 1: $D_2 = \{ e, a, b, c \}$

3 subgroups: $\{ e, a \}$, $\{ e, b \}$, $\{ e, c \}$

Example 2: $D_3 \approx S_3 \approx \{ e, a, b = a^{-1}, c = c^{-1}, d = d^{-1}, f = f^{-1} \}$

4 subgroups: $\{ e, a, b \}$, $\{ e, c \}$, $\{ e, d \}$, $\{ e, f \}$

Infinite Group : Group order = ∞

E.g. $T^d = \{ T(n) \mid n \in Z \}$

Some subgroups:

$$T_m^d = \left\{ T(mn) \,\middle|\, n \in \mathbf{Z} \right\}$$

Continuous Group : Elements specified by continuous parameters

E.g. Continuous translations T

Continuous rotations R(2), R(3)

Continuous translations & rotations E(2), E(3)

Crystallographic Point Groups:

C_n, C_{nv}, C_{nh},

D_n, D_{nv}, D_{nh}, D_{nd},

S_n,

T, T_d, T_h, (Tetrahedral)

O, O_h, (Cubic)

I (icosahedral)

n = 2,3,4,6

v: vertical σ

h: horizontal σ

D_n: C_n with $C_2 \perp C_n$

d: vert σ between 2 C_2's

S_n: C_n with i

The crystal point group symmetries

Crystal class	*Schoenflies symbol*	*International notation*
Triclinic:	C_1	1
	S_2	$\bar{1}$
Monoclinic:	C_{1h}	m
	C_2	2
	C_{2h}	$2/m$
Orthorhombic:	C_{2v}	$mm2$
	D_2	222
	D_{2h}	mmm

Tetragonal:	C_4	4
	C_{4v}	$4mm$
	C_{4h}	$4/m$
	D_4	422
	D_{2d}	$\bar{4}2m$
	D_{4h}	$4/mmm$
	S_4	$\bar{4}$
Rhombohedral:	C_3	3
	C_{3v}	$3m$
	D_3	32
	D_{3d}	$\bar{3}m$
	S_6	$\bar{3}$
Hexagonal:	C_6	6
	C_{6v}	$6mm$
	C_{3h}	$\bar{6}$
	C_{6h}	$6/m$
	D_6	622
	D_{3h}	$\bar{6}m2$
	D_{6h}	$6/mmm$
Cubic:	T	23
	T_d	$\bar{4}3m$
	T_h	$m3$
	O	432
	O_h	$m3m$

Matrix / Classical groups:

- General linear group GL(n)
  - Unitary group U(n)
    - Special Unitary group SU(n)
  - Orthogonal group O(n)
    - Special Orthogonal group SO(n)

2.3. The Rearrangement Lemma & the Symmetric Group

Lemma: Rearrangement

$p\,b = p\,c \;\rightarrow\; b = c$ where $p, b, c \in G$

Proof: p^{-1} both sides

Corollary: p G = G rearranged; likewise G p

Permutation: $p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}$

$p_i \leftarrow i$ (Active point of view)

Product: $p\,q = (p_k \leftarrow k) (q_i \leftarrow i)$ $= (p_{q_i} \leftarrow q_i)(q_i \leftarrow i) = (p_{q_i} \leftarrow i)$

$$pq = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ q_1 & q_2 & q_3 & \cdots & q_n \end{pmatrix}$$

$$= \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_n \\ p_{q_1} & p_{q_2} & p_{q_3} & \cdots & p_{q_n} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ q_1 & q_2 & q_3 & \cdots & q_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_{q_1} & p_{q_2} & p_{q_3} & \cdots & p_{q_n} \end{pmatrix}$$

(Rearranged)

$$pq = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_{q_1} & p_{q_2} & p_{q_3} & \cdots & p_{q_n} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ (pq)_1 & (pq)_2 & (pq)_3 & \cdots & (pq)_n \end{pmatrix}$$

$$(pq)_j = p_{q_j}$$

Identity:

$$e = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

Inverse:

$$p^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ (p^{-1})_1 & (p^{-1})_2 & (p^{-1})_3 & \cdots & (p^{-1})_n \end{pmatrix}$$

$$= \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

$i \leftarrow p_i$

Symmetric (Permutation) group $S_n \equiv$ { n! permutations of n objects }

n-Cycle = ($p_1, p_2, p_3, \ldots, p_n$) $\equiv \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_3 & p_4 & \cdots & p_1 \end{pmatrix}$

Every permutation can be written as a product of cycles

Example

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3)$$

$$q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)(2)$$

$$pq = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$$qp = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

$$p^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = p$$

$$q^{-1} = q$$

$$(pq)^{-1} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) = qp$$

Definition 2.5: Isomorphism

2 groups G & G ' are isomorphic (G ≈ G ') , if ⅄ a 1-1 onto mapping

ϕ : G → G ' g_i ⊏ ¬⅄ g_i' ∋ $g_i\, g_j = g_k$ ↔ $g_i\, g_j' = g_k'$

Examples:

- Rotational group C_n ≈ cyclic group C_n
- $D_3 \approx C_{3v} \approx S_3$

Theorem 2.1: Cayley

Every group of finite order n is isomorphic to a subgroup of S_n

Proof: Let $G = \{ g_1, g_2, \ldots, g_n \}$. The required mapping is

ϕ : G → S_n

$$g_j \mapsto p_j = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

where $g_j g_k = g_{j_k}$

$$\Rightarrow \quad g_j g_k = g_{j_k} \mapsto p_j p_k = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_{k_1} & j_{k_2} & \cdots & j_{k_n} \end{pmatrix}$$

$$= p_{j_k}$$

Example 1: $C_3 = \{ e, a, b = a^2 ; a^3 = e \} = \{ g_1, g_2, g_3 \}$

e	a	b
a	b	e
b	e	a

1	2	3
2	3	1
3	1	2

$$e \mapsto p_e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$$

$$a \mapsto p_a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

$$b \mapsto p_b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$C_3 \approx \{ e, (123), (321) \}$, subgroup of S_3

Example 2: $D_2 = \{ e, a = a^{-1}, b = b^{-1}, c = a\,b \}$

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$$e \mapsto p_e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4)$$

$$a \mapsto p_a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$$

$$b \mapsto p_b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

$$c \mapsto p_c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$$

$D_2 \approx \{ e, (12)(34), (13)(24), (14)(23) \}$, subgroup of S_4

Example 3: $C_4 = \{ e = a^4, a, a^2, a^3 \}$

e	a	a^2	a^3
a	a^2	a^3	e
a^2	a^3	e	a
a^3	e	a	a^2

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

$$e \to p_e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4)$$

$$a \to p_a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234)$$

$$a^2 \to p_{a^2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

$$a^3 \to p_{a^3} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1432)$$

$D_2 \approx$ { e, (1234), (13)(24), (1432) }, subgroup of S_4

Let S be a subgroup of S_n that is isomorphic to a group G of order n. Then

- The only element in S that contains 1-cycles is e (else, rearrangement therem is violated)
- All cycles in a given element are of the same length (else, some power of it will contain 1-cycles)

  E.g., $[(12)(345)]^2 = (1)(2)(345)^2$
- If order of G is prime, then S can contain only full n-cycles, ie, S is cyclic

Theorem 2.2: A group of prime order is isomorphic to C_n

Only 1 group for each prime order

Definition 2.6: Conjugate Elements

Let a , b ∈ G. b is conjugate to a, or b~a, if

$\exists\ p \in G \quad \ni \quad b = p\, a\, p^{-1}$

Example: S_3

- (12) ~ (31) since (23) (31) $(23)^{-1}$ = (23) (132) = (12)(3) = (12)
- (123) ~ (321) since (12) (321) (12) = (12) (1)(23) = (123)

Exercise: Show that for p, q ∈ S_n ,

$$p\, q\, p^{-1} = \begin{pmatrix} p_1 & \cdots & p_n \\ p_{q_1} & \cdots & p_{q_n} \end{pmatrix}$$

Hint:

$$p = \begin{pmatrix} 1 & \cdots & n \\ p_1 & \cdots & p_n \end{pmatrix} = \begin{pmatrix} q_1 & \cdots & q_n \\ p_{q_1} & \cdots & p_{q_n} \end{pmatrix}$$

Def: ~ is an equivalence relation if

- a ~ a (reflexive)
- a~b → b~a (symmetric)
- a~b, b~c → a~c (transitive)

Conjugacy is an equivalence relation

Proof :

$$a = e a e^{-1} \rightarrow a \sim a$$ (reflexive)

$$a \sim b \rightarrow \exists p \in G \ni a = p b p^{-1}$$

$$\therefore \quad b = p^{-1} a p = q b q^{-1} \qquad q = p^{-1} \in G \rightarrow b \sim a$$ (symmetric)

$$a \sim b, b \sim c \rightarrow \exists p, q \in G \ni a = p b p^{-1}, \ b = q c q^{-1}$$

$$\therefore \quad a = p q c q^{-1} p^{-1} = r c r^{-1} \qquad r = p q \in G \rightarrow a \sim c$$ (transitive)

An equivalence relation partitions (classifies members of) a set.

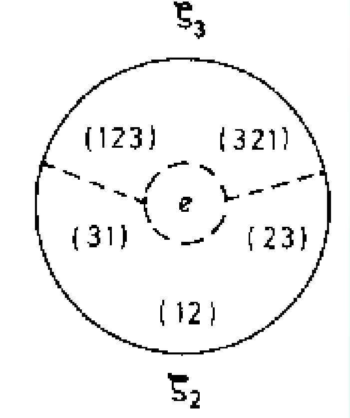
Definition 2.7: Conjugate Class

Let $a \in G$, the conjugate class of a is the set $\xi = \{ p\, a\, p^{-1} \mid p \in G \}$

Comments:

- Members of a class are equivalent & mutually conjugate
- Every group element belongs to 1 & only 1 class
- e is always a class by itself
- For matrix groups, conjugacy = similarity transform

Example 1: S_3 (3 classes):



- $\xi_1 = \{ e \}$ identity
- $\xi_2 = \{ (12), (23), (31) \}$ 2-cycles
- $\xi_3 = \{ (123), (321) \}$ 3-cycles

$q\,p\,q^{-1} = \left(q_i \to q_{p_i} \right)$ → Permutations with the same cycle structure belong to the same class.

Example 2: R(3) (Infinitely many classes):

Let $R_{\mathbf{u}}(\psi)$ be a rotation about **u** by angle ψ. **u** = unit vector

$R\,R_{\mathbf{u}}(\psi)\,R^{-1} = R_{R\mathbf{u}}(\psi)$ ® Class: $\xi(\psi) = \{ R_{\mathbf{u}}(\psi) ; \text{all } \mathbf{u} \}$

$\forall\ R \in R(3)$ = { All rotations of angle ψ }

Example 3: E_3 (Infinitely many classes):

Let $T_{\mathbf{u}}(b)$ be a translation along **u** by distance b.

$R\,T_{\mathbf{u}}(b)\,R^{-1} = T_{R\mathbf{u}}(b)$ ® Class: $\xi(b) = \{ T_{\mathbf{u}}(b) ; \text{all } \mathbf{u} \}$

$\forall\ R \in E_3$ = { All translations of distance b }

Def: Conjugate Subgroup

Let H be a subgroup of G & $a \in G$.

$H' = \{ a\, h\, a^{-1} \mid h \in H \}$ = Subgroup conjugate to H

Exercise:

- Show that H' is a subgroup of G
- Show that either H [illegible] H' or $H \cap H' = e$

Definition 2.8: Invariant Subgroup

H is an invariant subgroup of G if it is identical to all its conjugate subgroups.

i.e., $H = \{ a\, h\, a^{-1} \mid h \in H \} \quad \forall\ a \in G$

Examples:

- $\{ e, a^2 \}$ is an invariant subgroup of $C_4 = \{ e = a^4, a, a^2, a^3 \}$
- { e, (123), (321) } is an invariant subgroup of S_3 but { e, (12) } isn't
- $T^d{}_m$ is an invariant subgroup of T^d

Comments:

- An invariant subgroup must consist of entire classes
- Every group G has 2 trivial invariant subgroups {e} & G
- Existence of non-trivial invariant subgroup $\rightarrow$ G can be factorized

Definition 2.9: Simple & Semi-Simple Groups

A group is simple if it has no non-trivial invariant subgroup.

A group is semi-simple if it has no Abelian invariant subgroup.

Examples:

- C_n with n prime are simple.
- C_n with n non-prime are neither simple nor semi-simple.

  $n = p\,q \rightarrow \{ e, C_p, C_{2p}, \dots, C_{(q-1)\,p} \}$ is an Abelian invariant subgroup
- S_3 is neither simple nor semi-simple. { e, (123), (321) } is spoiler.
- SO(3) is simple but SO(2) is not. Spoilers: C_n

2.5 Cosets and Factor (Quotient) Groups

Definition 2.10: Cosets

Let $H = \{ h_1, h_2, \dots \}$ be a subgroup of *G* & $p \in G - H$.

Then $p H = \{ p h_1, p h_2, \dots \}$ is a left coset of H,

& $H p = \{ h_1 p, h_2 p, \dots \}$ is a right coset of H.

- Neither p H, nor H p, is a subgroup of G (no e)
- All cosets of H have the same order as H (rearrangement theorem)

Lemma: Either $p H = q H$ or $p H \cap q H = \varnothing$

Proof:

If $\exists\ h_i$ & h_j $\ni$ $p h_i = q h_j \rightarrow p = q h_j h_i^{-1} = q h_k \in qH$

\ $p H = q h_k H = q H$

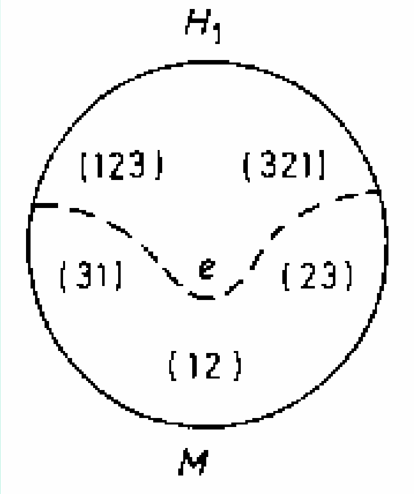
Negation of above gives 2nd part of lemma.

Corollary: G is partitioned by cosets of H. $\rightarrow$ Lagrange theorem

Theorem 2.3: Lagrange (for finite groups)

H is a subgroup of G $\rightarrow$ Order(G) / Order(H) $= n_G / n_H \in N$

e	(123)	(132)	(23)	(13)	(12)
(123)	(132)	e	(12)	(23)	(13)
(132)	e	(123)	(13)	(12)	(23)
(23)	(13)	(12)	e	(123)	(132)
(13)	(12)	(23)	(132)	e	(123)
(12)	(23)	(13)	(123)	(132)	e

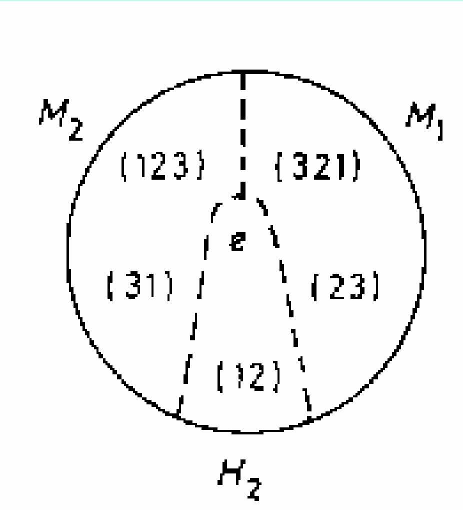


Examples: S_3

- H_1 = { e, (123), (321) }. One coset:

  $M = (12) H_1 = (23) H_1 = (31) H_1$

  = { (12), (23), (31) }



- H_2 = { e, (12) } . Two cosets:

  $M_1 = (23) H_2 = (321) H_2$ = { (23), (321) }

  $M_2 = (31) H_2 = (123) H_2$ = { (31), (123) } Thm: H is an invariant subgroup $\rightarrow$ pH = Hp

Proof: H invariant $\rightarrow$ $pHp^{-1} = H$

Theorem 2.4: Factor / Quotient Group G/H

Let H be an invariant subgroup of G. Then

$G/H \equiv \{ \{ pH \mid p \in G \}, \bullet \}$ with $pH \bullet qH \equiv (pq)\, H$

is a (factor) group of G. Its order is n_G / n_H.

Example 1: $C_4 = \{ e = a^4, a, a^2, a^3 \}$

$H = \{ e, a^2 \}$ is an invariant subgroup.

Coset $M = a\,H = a^2\,H = \{ a, a^3 \}$.

Factor group $C_4/H = \{ H, M \} \approx C_2$

H	M
M	H

Example 2: S_3 = { e, (123), (132), (23), (13), (12) }

e	(123)	(132)	(23)	(13)	(12)
(123)	(132)	e	(12)	(23)	(13)
(132)	e	(123)	(13)	(12)	(23)
(23)	(13)	(12)	e	(123)	(132)
(13)	(12)	(23)	(132)	e	(123)
(12)	(23)	(13)	(123)	(132)	e

H = { e, (123), (132) } is invariant

Coset M = { (23), (13), (12) }

Factor group S_3/H = { H, M } $\approx C_2$

$C_{3v} / C_3 \approx C_2$

Example 3: $T^d \equiv \Gamma = \{ T(n), n \in Z \}$

$\Gamma_m = \{ T(mn), n \in Z \}$ is an invariant subgroup.

Cosets: $T(k)\, \Gamma_m$ $k = 1, \ldots, m-1$ & $T(m)\, \Gamma_m = \Gamma_m$

Products: $T(k)\, \Gamma_m \bullet T(j)\, \Gamma_m = T(k+j)\, \Gamma_m$

Factor group: $\Gamma / \Gamma_m = \{ \{ T(k)\, \Gamma_m \mid k = 1, \ldots, m-1 \}, \bullet \} \approx C_m$

Caution: $\Gamma_m \approx \Gamma$

Example 4: E_3

H = T(3) is invariant. $E_3 / T(3) \approx R(3)$

2.6 Homomorphisms

Definition 2.11: Homomorphism

G is homomorphic to G' (G ~ G') if ㅅ a group structure preserving mapping from G to G', i.e.

ㅅ ϕ : G → G' g ㄱㅅ g' = ϕ(g)

϶ a b = c ㄱㅅ a' b' = c'

Isomomorphism: ϕ is invertible (1-1 onto).

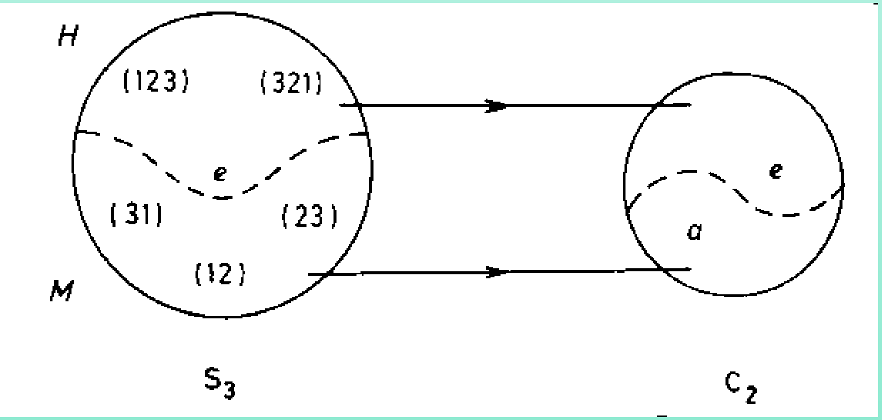
Example:

ϕ: $S_3 \to C_2$

withϕ(e) = ϕ[(123)] = ϕ[(321)] = e

ϕ[(23)] = ϕ[(31)] = ϕ[(12)] = a

is a homorphism $S_3 \sim C_2$.



Theorem 2.5:

Let ϕ: G $\rightarrow$ G' be a homomorphism and **Kernel** = K = { g | ϕ(g) = e' }

Then K is an invariant subgroup of G and G/K $\approx$ G'

Proof 1 (K is a subgroup of G):

ϕ is a homomorphism:

$\therefore$ a, b $\in$ K $\rightarrow$ $\phi(ab) = \phi(a)\,\phi(b) = e'\,e' = e'$ $\rightarrow$ ab $\in$ K (closure)

$\phi(ae) = \phi(a)\,\phi(e) = e'\,\phi(e) = \phi(e)$

$= \phi(a) = e'$ $\rightarrow$ $\phi(e) = e'$ $\rightarrow$ e $\in$ K (identity)

$\phi(a^{-1}a) = \phi(a^{-1})\,\phi(a) = \phi(a^{-1})\,e' = \phi(a^{-1})$

$= \phi(e) = e'$ $\rightarrow$ $a^{-1} \in K$ (inverse)

Associativity is automatic. QED

Proof 2 (K is a invariant):

Let $a \in K$ & $g \in G$.

$$\phi(g\, a\, g^{-1}) = \phi(g)\, \phi(a)\, \phi(g^{-1}) = \phi(g)\, \phi(g^{-1}) = \phi(g\, g^{-1}) = \phi(e) = e'$$

$\rightarrow \quad g\, a\, g^{-1} \in K$

Proof 3 ($G/K \approx G'$):

$$G/K = \{ pK \mid p \in G \}$$

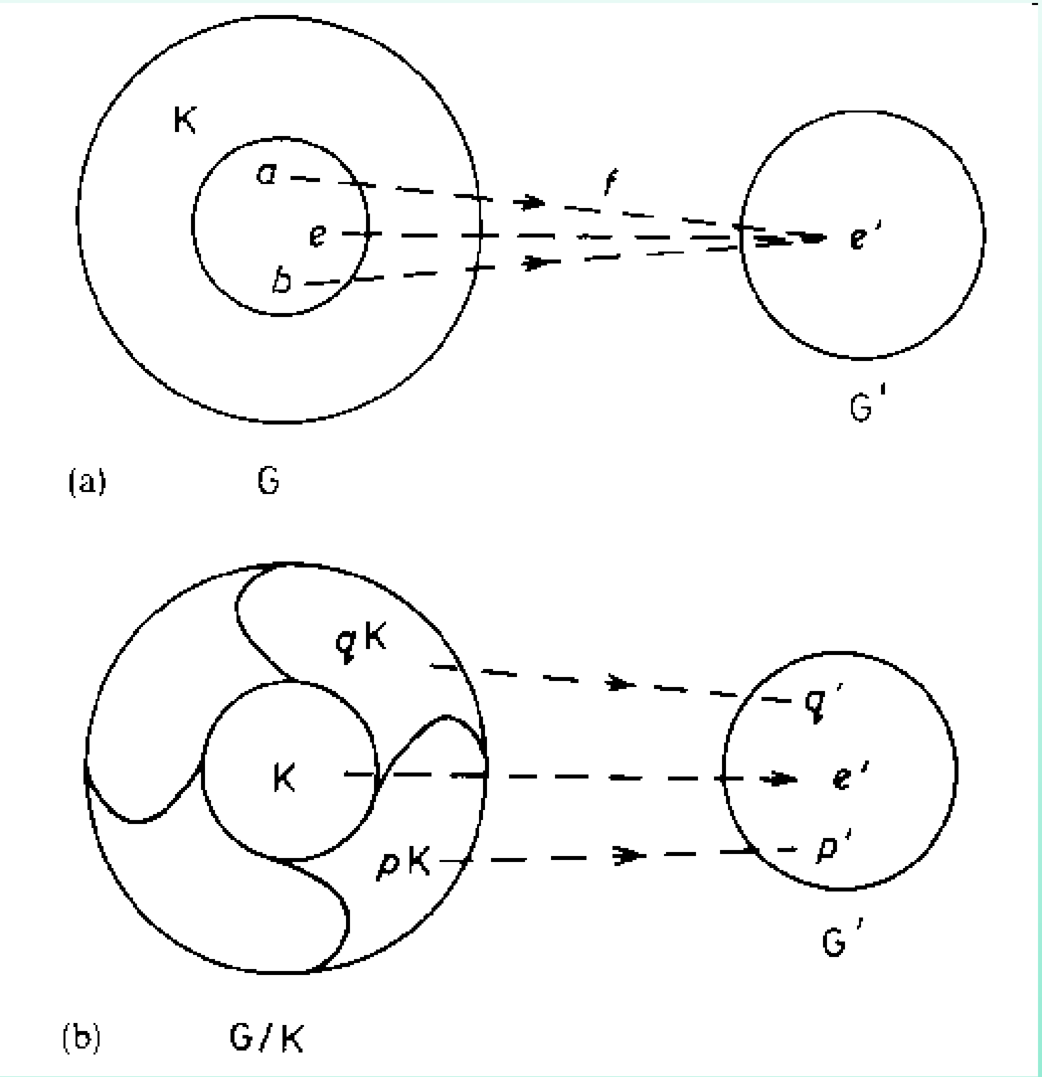
$\therefore \quad \phi(pa) = \phi(p)\, \phi(a) = \phi(p)\, e' = \phi(p) \qquad \forall \quad a \in K$

i.e., ϕ maps the entire coset pK to one element $\phi(p)$ in G'.

Hence, $\psi : G/K \rightarrow G'$ with $\psi(pK) = \phi(p) = \phi(q \in pK)$ is 1-1 onto.

$$\psi(pK\, qK) = \psi[(pq)K] = \phi(pq) = \phi(p)\, \phi(q) = \psi(pK)\, \psi(qK)$$

$\rightarrow \psi$ is a homomorphism. QED



Kernel

G/K ≈ G'

2.7 Direct Products

Definition 2.12: Direct Product Group $A \otimes B$

Let A & B be subgroups of group G such that

- $a\, b = b\, a \qquad \forall \quad a \in A \quad \& \quad b \in B$
- $\forall \; g \in G, \; \exists! \; a \in A \;\; \& \;\; b \in B \;\; \ni \;\; g = a\, b = b\, a$

Then G is the direct product of A & B, i.e, $G = A \otimes B = B \otimes A$

Example 1: $C_6 = \{ e = a^6, a, a^2, a^3, a^4, a^5 \}$

Let $A = \{ e, a^3 \}$ & $B = \{ e, a^2, a^4 \}$

- $a\, b = b\, a$ trivial since C_6 is Abelian
- $e = e\, e, \;\; a = a^3 a^4, \;\; a^2 = e\, a^2, \;\; a^3 = a^3 e, \;\; a^4 = e\, a^4, \;\; a^5 = a^3 a^2$

$\therefore \quad C_6 = A \otimes B \approx C_2 \otimes C_3$

Example 2: $O(3) = R(3) \otimes \{ e, I_S \}$

Thm:

$G = A \otimes B \quad \rightarrow$

- A & B are invariant subgroups of G
- $G/A \approx B, \quad G/B \approx A$

Proof:

$g = a\, b \quad \rightarrow \quad g\, a'\, g^{-1} = a\, b\, a'\, b^{-1}\, a^{-1} = a\, a'\, b\, b^{-1}\, a^{-1} = a\, a'\, a^{-1} \in A$

∴ A is invariant ; dido B.

$G = \{ a\, B \mid a \in A \} \rightarrow \quad G/B \approx A$ & similarly for B

Caution: $G/B \approx A$ does not imply $G = A \otimes B$

Example: S_3

$H = \{ e, \{123\}, \{321\} \}$ is invariant. Let $H_i = \{ e, (j\,k) \}$ (i,j,k cyclic)

Then $S_3/H \approx H_i$ but $S_3 \neq H \otimes H_i$