LIMITS AND CONTINUITY

THE LIMIT PROCESS (AN INTUITIVE INTRODUCTION)

We could begin by saying that limits are important in calculus, but that would be a major understatement. *Without limits, calculus would not exist. Every single notion of calculus is a limit in one sense or another*.

For example:

What is the slope of a curve? It is the limit of slopes of secant lines.

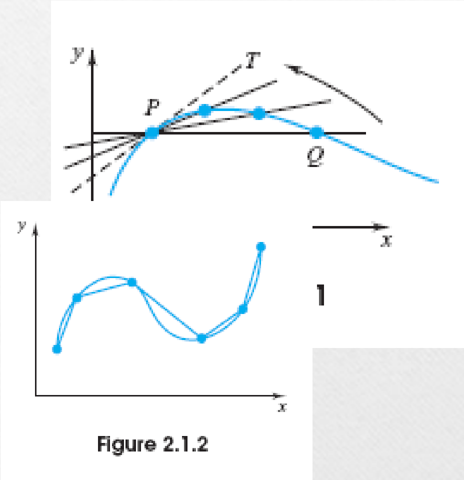

Figure 2.1.2

What is the length of a curve? It is the limit of the lengths of polygonal paths inscribed in the curve.

What is the area of a region bounded by a curve? It is the limit of the sum of areas of approximating rectangles.

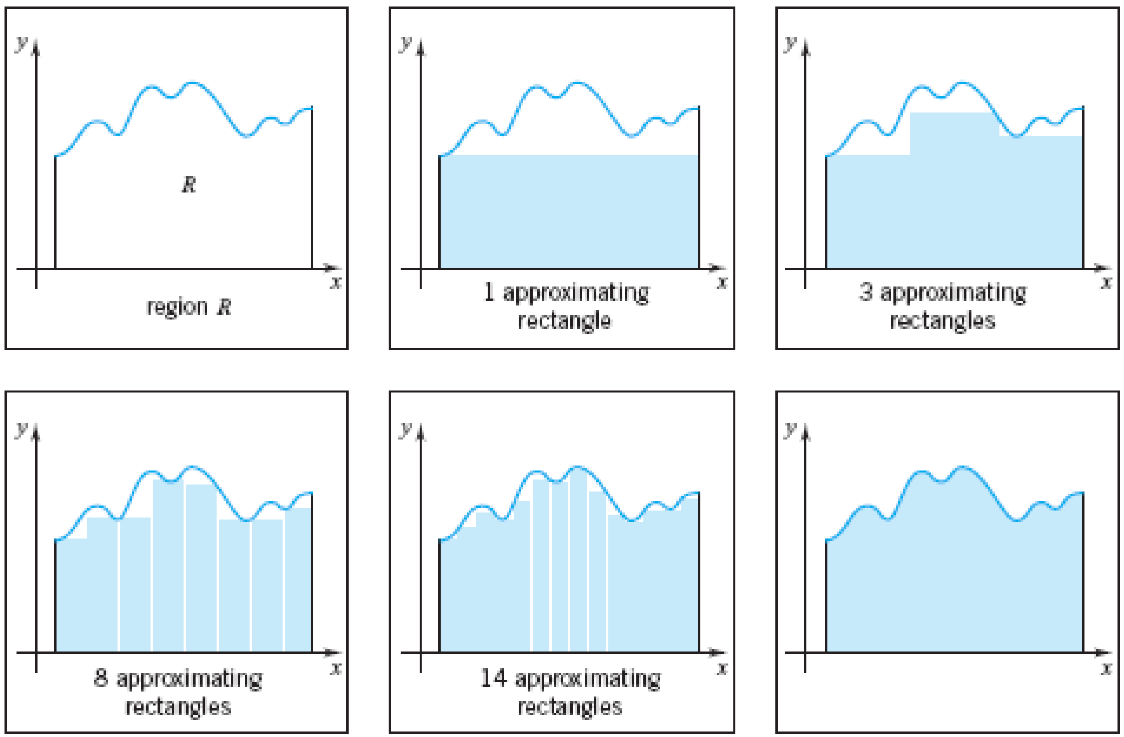

Figure 2.1.3

The Idea of a Limit

We start with a number c and a function f defined at all numbers x near c but not necessarily at c itself. In any case, whether or not f is defined at c and, if so, how is totally irrelevant.

Now let L be some real number. We say that *the limit of* $f(x)$ *as* x *tends to* c *is* L and write

$$\lim_{x \to c} f(x) = L$$

provided that (roughly speaking)

as x approaches c, f(x) approaches L

or (somewhat more precisely) provided that

$f(x)$ *is close to* L *for all* $x \neq c$ *which are close to* c.

Example 1

Set $f(x) = 4x + 5$ and take $c = 2$. As x approaches 2, $4x$ approaches 8 and $4x + 5$ approaches $8 + 5 = 13$. We conclude that

$$\lim_{x \to 2} f(x) = 13.$$

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Example 2

Set

$$f(x) = \sqrt{1-x} \quad \text{and take } c = -8.$$

As x approaches -8, $1 - x$ approaches 9 and $\sqrt{1-x}$ approaches 3. We conclude that

$$\lim_{x \to -8} f(x) = 3$$

If for that same function we try to calculate

$$\lim_{x \to 2} f(x)$$

we run into a problem. The function $f(x) = \sqrt{1-x}$ is defined only for $x \leq 1$. It is therefore not defined for x near 2, and the idea of taking the limit as x approaches 2 makes no sense at all:

$$\lim_{x \to 2} f(x) \quad \text{does not exist.}$$

Example 3

$$\lim_{x\to 3}\frac{x^3-2x+4}{x^2+1}=\frac{5}{2}.$$

First we work the numerator: as x approaches 3, x^3 approaches 27, $-2x$ approaches –6, and $x^3 - 2x + 4$ approaches 27 – 6 + 4 = 25. Now for the denominator: as x approaches 3, $x^2 + 1$ approaches 10. The quotient (it would seem) approaches 25/10 = 52.

The curve in Figure 2.1.4 represents the graph of a function f. The number c is on the x-axis and the limit L is on the y-axis. As x approaches c along the x-axis, $f(x)$ approaches L along the y-axis.

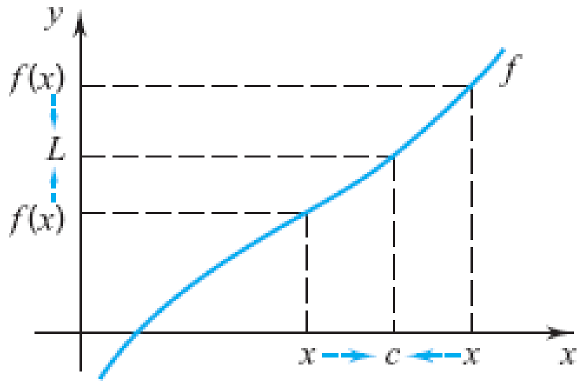

Figure 2.1.4

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As we have tried to emphasize, in taking the limit of a function f as x tends to c, it does not matter whether f is defined at c and, if so, how it is defined there. The only thing that matters is the values taken on by f at numbers x near c. Take a look at the three cases depicted in Figure 2.1.5. In the first case, $f(c) = L$. In the second case, f is not defined at c. In the third case, f is defined at c, but $f(c) \neq L$. However, in each case

$$\lim_{x \to c} f(x) = L$$

because, as suggested in the figures,

as x approaches c, $f(x)$ approaches L.

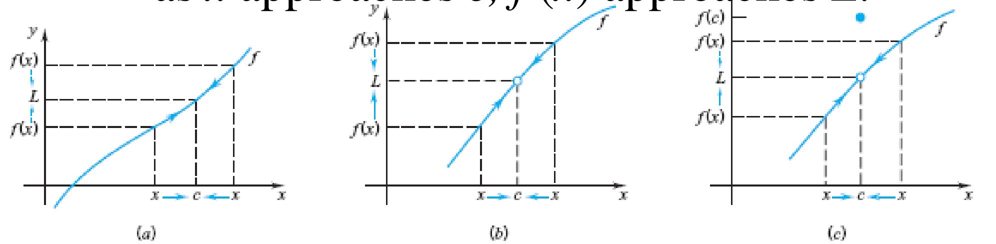

Figure 2.1.5

Example 4

Set $f(x) = \dfrac{x^2 - 9}{x - 3}$

and let c = 3. Note that the function f is not defined at 3: at 3, both numerator and denominator are 0. But that doesn't matter. For $x \neq 3$, and therefore *for all x near* 3,

$$\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3$$

Therefore, if x is close to 3, then $\dfrac{x^2 - 9}{x - 3} = x + 3$

is close to 3 + 3 = 6. We conclude that

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3) = 6$$

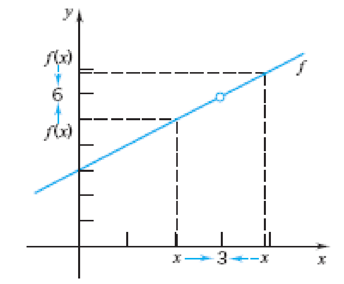

Figure 2.1.6

Example 5

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 12.$$

The function $f(x) = \dfrac{x^3 - 8}{x - 2}$ is undefined at $x = 2$. But, as we said before, that doesn't matter. For all $x \neq 2$,

$$\frac{x^3 - 8}{x - 2} = \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = x^2 + 2x + 4.$$

Therefore,

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 4) = 12.$$

Example 6

If $f(x) = \begin{cases} 3x - 4, & x \neq 0 \\ 10, & x \neq 0, \end{cases}$ then $\lim_{x \to 0} f(x) = -4$.

It does not matter that f(0) = 10. For x ≠ 0, and thus for all x near 0,

$f(x) = 3x - 4$ and therefore $\lim_{x \to 0} f(x) = \lim_{x \to 0} (3x - 4) = -4$.

One-Sided Limits

Numbers x near c fall into two natural categories: those that lie to the left of c and those that lie to the right of c. We write

$$\lim_{x \to c^-} f(x) = L$$

[The left-hand limit of *f(x)* as x tends to c is L.]

to indicate that

as x approaches c from the left, f(x) approaches L.

We write

$$\lim_{x \to c^+} f(x) = L$$

[The right-hand limit of *f(x)* as x tends to c is L.]

to indicate that

as x approaches c from the right, f(x) approaches L

Example

Take the function indicated in Figure 2.1.7. As x approaches 5 from the left, $f(x)$ approaches 2; therefore

$$\lim_{x \to 5^-} f(x) = 2$$

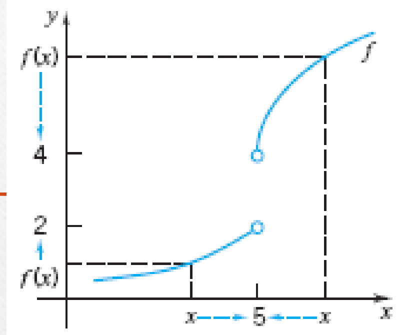

Figure 2.1.7

As x approaches 5 from the right, $f(x)$ approaches 4; therefore

$$\lim_{x \to 5^+} f(x) = 4$$

The full limit, $\lim_{x \to 5} f(x)$, does not exist: consideration of $x < 5$ would force the limit to be 2, but consideration of $x > 5$ would force the limit to be 4.

For a full limit to exist, both one-sided limits have to exist and they have to be equal.

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Example 7

For the function *f* indicated in figure 2.1.8,

$$\lim_{x\to(-2)^-} f(x)=5 \qquad \text{and} \qquad \lim_{x\to(-2)^+} f(x)=5$$

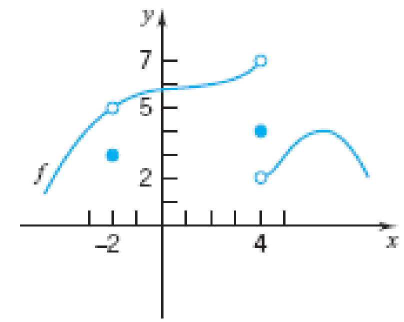

Figure 2.1.8

In this case

$$\lim_{x\to-2} f(x)=5$$

It does not matter that $f(-2)=3$.

Examining the graph of *f* near $x = 4$, we find that

$$\lim_{x\to4^-} f(x)=7 \qquad \text{whereas} \qquad \lim_{x\to4^+} f(x)=2$$

Since these one-sided limits are different,

$$\lim_{x\to4} f(x) \qquad \text{does not exist.}$$

Example 8

Set $f(x) = x/|x|$. Note that $f(x) = 1$ for $x > 0$, and $f(x) = -1$ for $x < 0$:

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

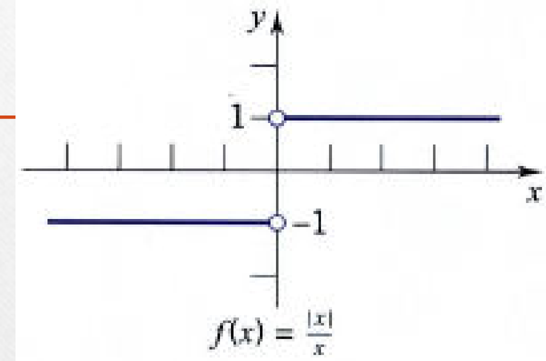

Figure 2.1.9

Let's try to apply the limit process at different numbers c.

If $c < 0$, then for all x sufficiently close to c, $x < 0$ and $f(x) = -1$. It follows that for $c < 0$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$$

If $c > 0$, then for all x sufficiently close to c, $x > 0$ and $f(x) = 1$. It follows that for $c < 0$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$$

However, the function has no limit as x tends to 0:

$$\lim_{x \to 0^-} f(x) = -1 \qquad \text{but} \qquad \lim_{x \to 0^+} f(x) = 1.$$

Example 9

We refer to function indicated in Figure 2.1.10 and examine the behavior of $f(x)$ for x close to 3 and close to to 7.

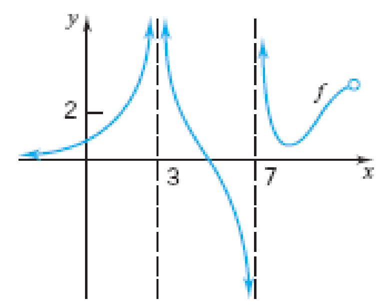

Figure 2.1.10

As x approaches 3 from the left or from the right, $f(x)$ becomes arbitrarily large and cannot stay close to any number L. Therefore

$$\lim_{x \to 3} f(x) \qquad \text{does not exist.}$$

As x approaches 7 from the left, $f(x)$ becomes arbitrarily large and cannot stay close to any number L. Therefore

$$\lim_{x \to 7} f(x) \qquad \text{does not exist.}$$

The same conclusion can be reached by noting as x approaches 7 from the right, $f(x)$ becomes arbitrarily large.

Remark To indicate that $f(x)$ becomes arbitrarily large, we can write $f(x) \to \infty$. To indicate that $f(x)$ becomes arbitrarily large negative, we can write $f(x) \to -\infty$.

Consider Figure 2.1.10, and note that for the function depicted there the following statements hold:

$$\text{as } x \to 3^{-}, \quad f(x) \to (\infty) \quad \text{and} \quad \text{as } x \to 3^{+}, \quad f(x) \to \infty.$$

Consequently,

$$\text{as } x \to 3, \quad f(x) \to \infty.$$

Also,

$$\text{as } x \to 7^{-}, \quad f(x) \to -\infty \quad \text{and} \quad \text{as } x \to 7^{+}, \quad f(x) \to \infty.$$

We can therefore write

$$\text{as } x \to 7, \quad |f(x)| \to \infty.$$

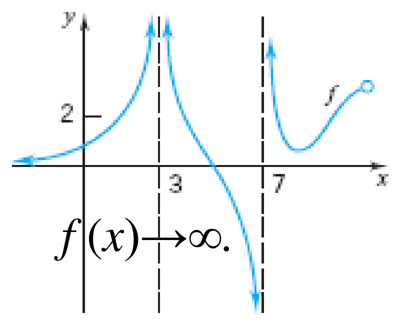

Figure 2.1.10

Example 10

We set

$$f(x) = \frac{1}{x - 2}$$

and examine the behavior of $f(x)$ (a) as x tends to 4 and then (b) as x tends to 2.

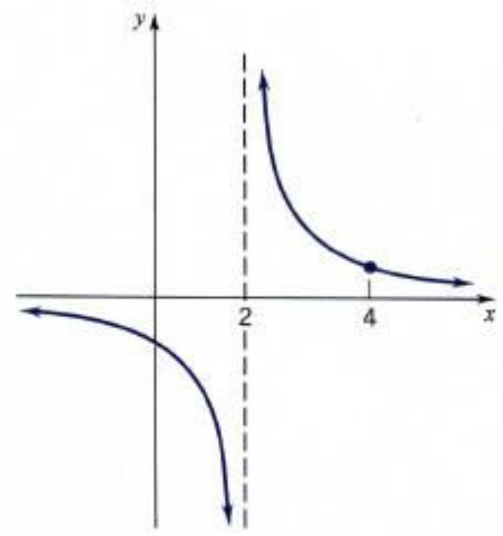

Figure 2.1.11

Example 11

$$\text{Set } f(x) = \begin{cases} 1 - x^2, & x < 1 \\ 1/(x-1), & x > 1. \end{cases}$$

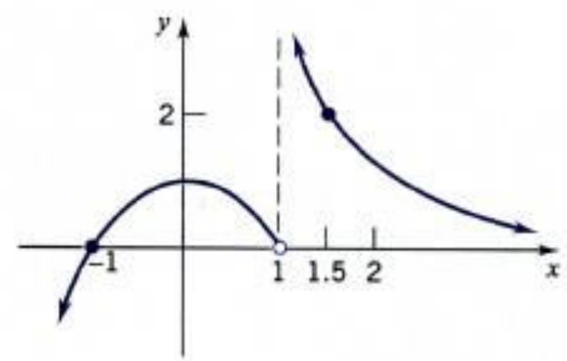

Figure 2.1.12

Example 12

Here we set $f(x) = \sin(\pi/x)$ and show that the function can have no limit as $x \to 0$

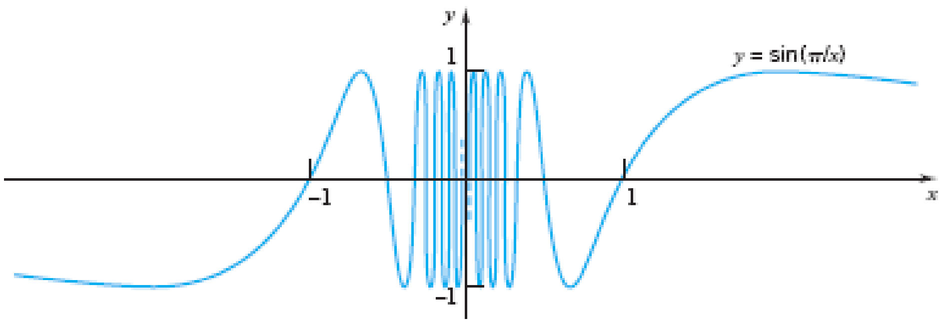

Figure 2.1.13

The function is not defined at $x = 0$, as you know, that's irrelevant. What keeps f from having a limit as $x \to 0$ is indicated in Figure 2.1.13. As $x \to 0$, $f(x)$ keeps oscillating between $y = 1$ and $y = -1$ and therefore cannot remain close to any one number L.

Example 13

Let $f(x) = (\sin x)/x$. If we try to evaluate f at 0, we get the meaningless ratio 0/0; f is not defined at x = 0. However, f is defined for all $x \neq 0$, and so we can consider

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

We select numbers that approach 0 closely from the left and numbers that approach 0 closely from the right. Using a calculator, we evaluate f at these numbers. The results are tabulated in Table 2.1.1.

Table 2.1.1

(Left side)		(Right side)	
x (radians)	$\frac{\sin x}{x}$	x (radians)	$\frac{\sin x}{x}$
−1	0.84147	1	0.84147
−0.5	0.95885	0.5	0.95885
−0.1	0.99833	0.1	0.99833
−0.01	0.99998	0.01	0.99998
−0.001	0.99999	0.001	0.99999

These calculations suggest that

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

and therefore that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

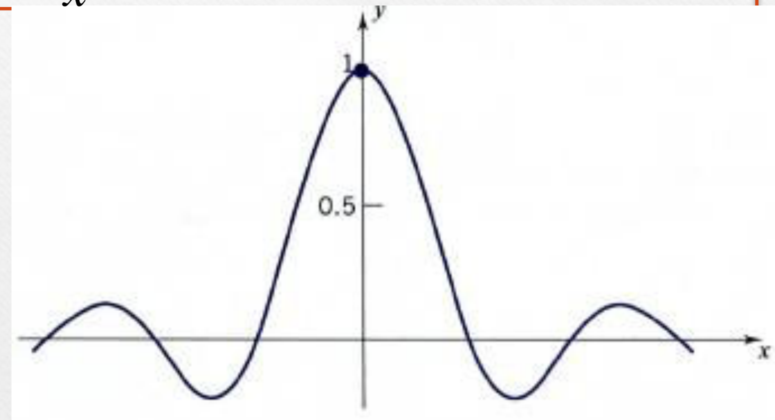

Figure 2.1.14

The graph of *f*, shown in Figure 2.1.14, supports this conclusion. A proof that this limit is indeed 1 is given in Section 2.5.

Summary of Limits That Fail to Exist

Examples 7-13 illustrate various ways in which the limit of a function f at a number c may fail to exist. We summarize the typical cases here:

(i) $\lim_{x \to c^-} f(x) = L_1, \quad \lim_{x \to c^+} f(x) = L_2$ and $L_1 \neq L_2$ (Examples 7, 8).
(The left-hand and right-hand limits of f at c each exist, but they are not equal.)

(ii) $f(x) \to \pm\infty$ as $x \to c^-$, or $f(x) \to \pm\infty$ as $x \to c^+$, or both (Examples 9, 10, 11). (The function f is unbounded as x approaches c from the left, or from the right, or both.)

(iii) $f(x)$ "oscillates" as $x \to c^-$, c^+ or c (Examples 12, 13).

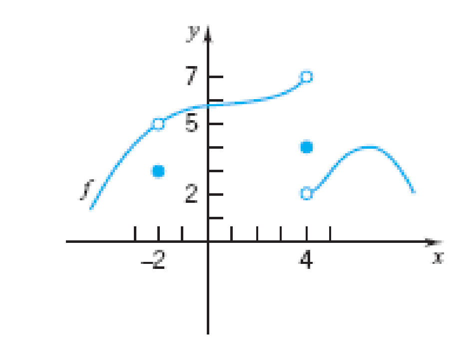

Figure 2.1.8

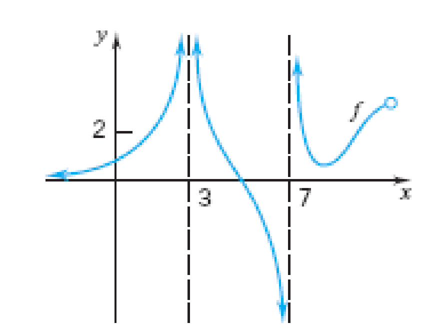

Figure 2.1.10

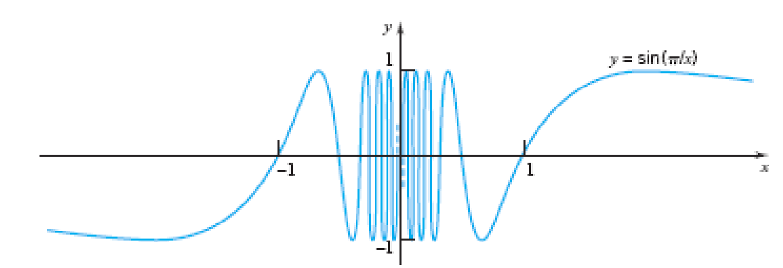

Figure 2.1.13

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DEFINITION 2.2.1 THE LIMIT OF A FUNCTION

Let f be a function defined at least on an open interval $(c - p, c + p)$ except possibly at c itself. We say that

$$\lim_{x \to c} f(x) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - c| < \delta, \qquad \text{then} \qquad |f(x) - L| < \epsilon.$$

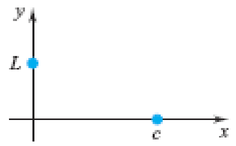

Figure 2.2.1

Figures 2.2.1 and 2.2.2 illustrate this definition.

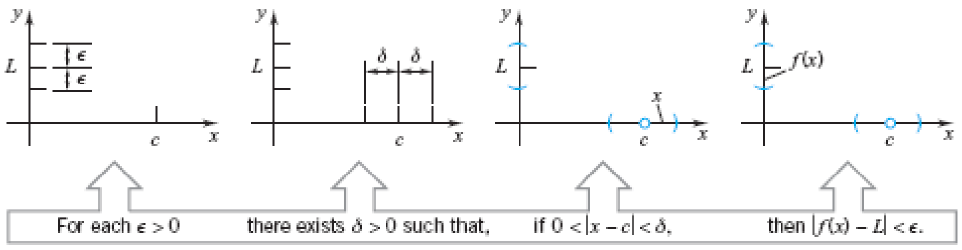

Figure 2.2.2

In Figure 2.2.3, we give two choices of ε and for each we display a suitable δ. For a δ to be suitable, all points within δ of c (with the possible exception of c itself) must be taken by the function f to within ε of L. In part (b) of the figure, we began with a smaller ε and had to use a smaller δ.

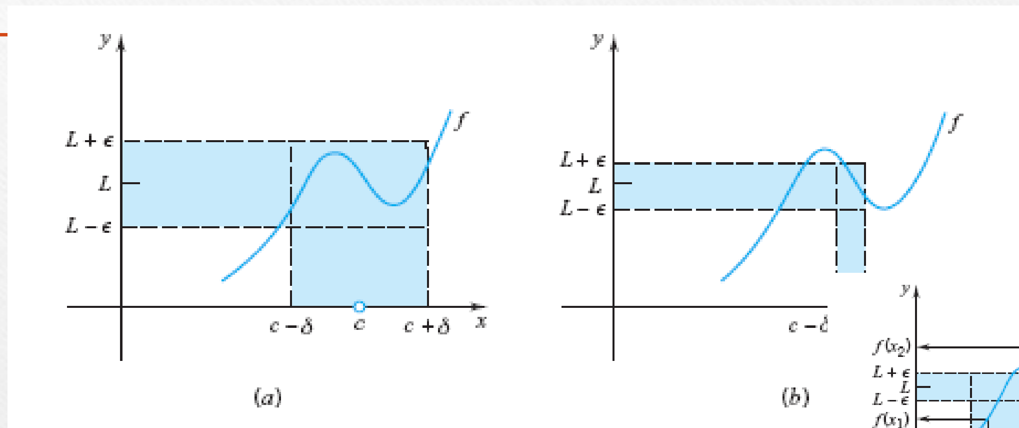

Figure 2.2.3

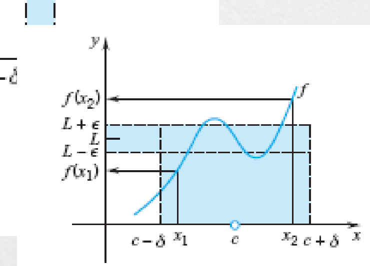

Figure 2.2.4

The δ of Figure 2.2.4 is too large for the given ε. In particular, the points marked x_1 and x_2 in the figure are not taken by f to within ε of L.

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(2.2.2) Let f be defined at least on an open interval $(c-p, c+p)$ except possibly at c itself. We say that

$$\lim_{x \to c} f(x) = L$$

if for each open interval $(L-\epsilon, L+\epsilon)$ there is an open interval $(c-\delta, c+\delta)$ such that all the numbers in $(c-\delta, c+\delta)$, with the possible exception of c itself, are mapped by f into $(L-\epsilon, L+\epsilon)$.

The limit process can be described entirely in terms of open intervals as shown in Figure 2.2.5.

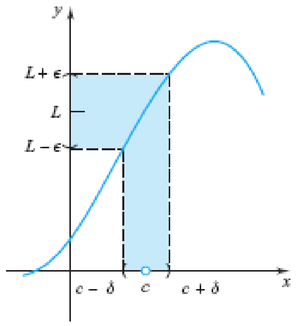

Figure 2.2.5

Example 1

Show that

$$\lim_{x \to 2} (2x - 1) = 3. \qquad \text{(Figure 2.2.6)}$$

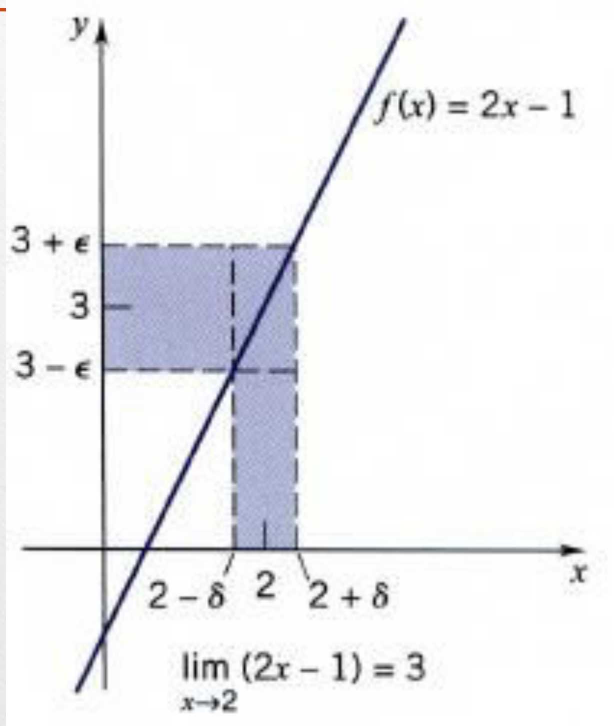

Figure 2.2.6

Example 2

Show that

$$\lim_{x \to -1} (2 - 3x) = 5. \qquad \text{(Figure 2.2.7)}$$

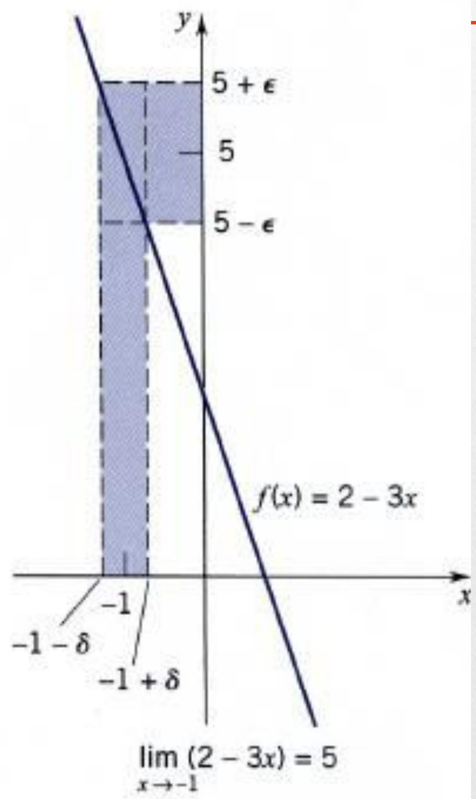

Figure 2.2.7

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Example 3

For each number c

(2.2.3)
$$\boxed{\lim_{x \to c} x = c.}$$

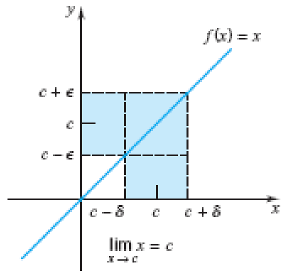

Figure 2.2.8

Example 4

For each real number c

(2.2.4)
$$\boxed{\lim_{x \to c} |x| = |c|.}$$

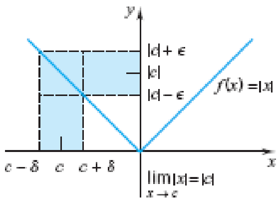

Figure 2.2.9

Example 5

For each constant k

(2.2.5)
$$\boxed{\lim_{x \to c} k = k.}$$

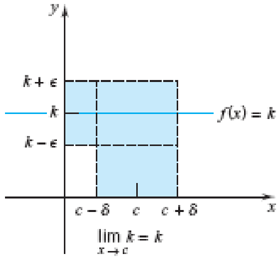

Figure 2.2.10

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Example 6
Show that

$$\lim_{x \to 3} x^2 = 9. \qquad \text{(Figure 2.2.11)}$$

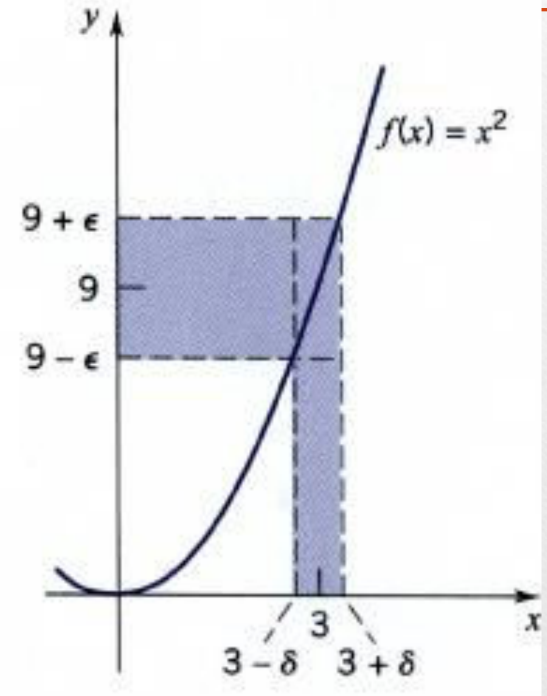

Figure 2.2.11

Example 7
Show that

$$\lim_{x \to 4} \sqrt{x} = 2.$$

(Figure 2.2.12)

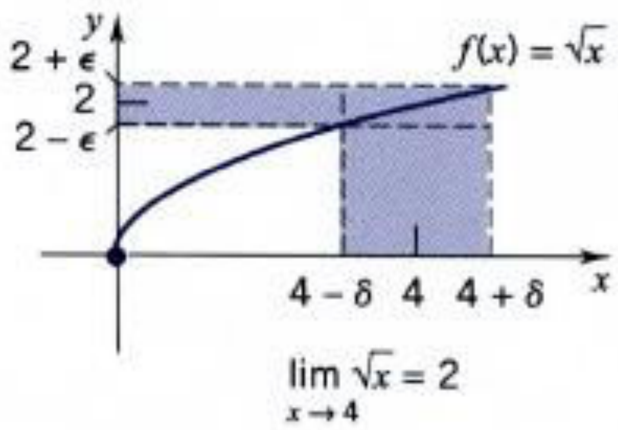

Figure 2.2.12

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There are several different ways of formulating the same limit statement. Sometimes one formulation is more convenient, sometimes another, In particular, it is useful to recognize that the following four statements are equivalent:

(2.2.6)

$$\boxed{\begin{array}{ll} \text{(i)}\ \lim_{x\to c} f(x) = L & \text{(ii)}\ \lim_{h\to 0} f(c+h) = L \\ \text{(iii)}\ \lim_{x\to c}(f(x) - L) = 0 & \text{(iv)}\ \lim_{x\to c} |f(x) - L| = 0. \end{array}}$$

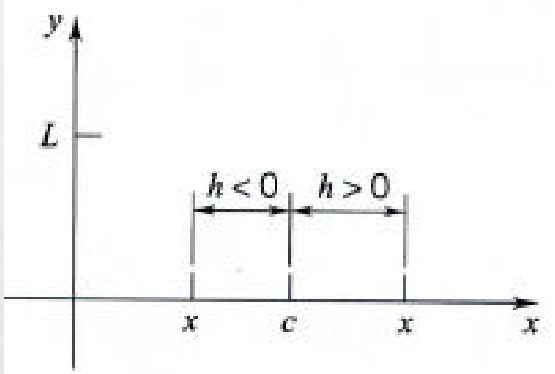

Figure 2.2.13

Example 8

For $f(x) = x^2$, we have

$$\lim_{x \to 3} x^2 = 9 \qquad \lim_{h \to 3} (3 + h)2 = 9$$

$$\lim_{x \to 3} (x^2 - 9) = 0 \qquad \lim_{x \to 3} | x^2 - 9 | = 0.$$

DEFINITION 2.2.7 LEFT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c - p, c)$. We say that

$$\lim_{x \to c^-} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if} \quad c - \delta < x < c, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

DEFINITION 2.2.8 RIGHT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c, c + p)$. We say that

$$\lim_{x \to c^+} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if} \quad c < x < c + \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

One-sided limits give us a simple way of determining whether or not a (two-sided) limit exists:

(2.2.9) $$\boxed{\lim_{x \to c} f(x) = L \quad \text{iff} \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.}$$

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Example 9

For the function defined by setting

$$f(x)=\begin{cases}2x+1, & x\le 0\\ x^2-x, & x>0\end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

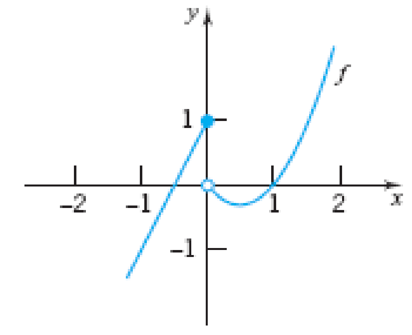

Figure 2.2.14

Proof

The left- and right-hand limits at 0 are as follows:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(2x+1)=1, \qquad \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x^2-x)=0$$

Since these one-sided limits are different,

$\lim_{x\to 0} f(x)$ does not exist.

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Example 10

For the function defined by setting

$$g(x) = \begin{cases} 1 + x^2, & x < 1 \\ 3, & x = 1 \\ 4 - 2x, & x > 1, \end{cases}$$

$\lim_{x \to 1} g(x) = 2.$

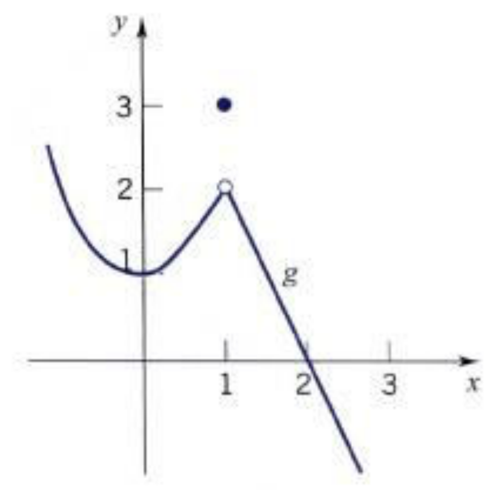

Figure 2.2.15

Proof

The left- and right-hand limits at 1 are as follows:

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (1 + x^2) = 2, \qquad \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (4 - 2x) = 2.$$

Thus, $\lim_{x \to 1} g(x) = 2$. NOTE: It does not matter that $g(1) \neq 2$.

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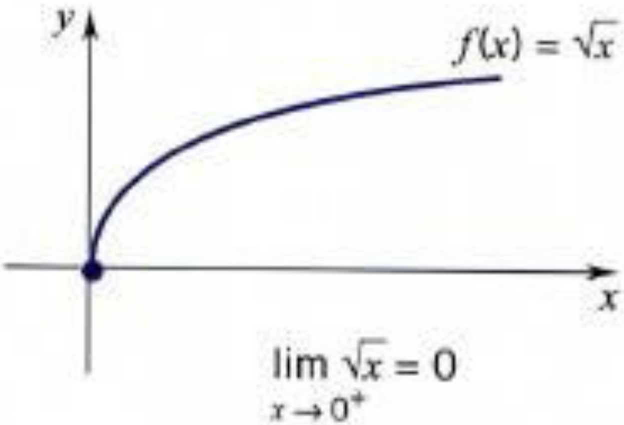

Figure 2.2.16

THEOREM 2.3.1 THE UNIQUENESS OF A LIMIT

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$, then $L = M$.

THEOREM 2.3.2

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(i) $\lim_{x\to c} [f(x) + g(x)] = L + M$,

(ii) $\lim_{x\to c} [\alpha f(x)] = \alpha L$ $|\alpha|$ a real number

(iii) $\lim_{x\to c} [f(x)g(x)] = LM$.

The following properties are extensions of Theorem 2.3.2.

(2.3.3) $$\lim_{x\to c}[f(x) - g(x)] = L - M$$

(2.3.4) $$\lim_{x\to c}[\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x)] = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_n L_n.$$

(2.3.5) $$\lim_{x\to c}[f_1(x) f_2(x) \cdots f_n(x)] = L_1 L_2 \cdots L_n.$$

(2.3.6) $$\lim_{x\to c} P(x) = P(c).$$

Examples

$$\lim_{x \to 1} (5x^2 - 12x + 2) = 5(1)^2 - 12(1) + 2 = -5,$$

$$\lim_{x \to 0} (14x^5 - 7x^2 + 2x + 8) = 14(0)^5 - 7(0)^2 + 2(0) + 8 = 8$$

$$\lim_{x \to -1} (2x^3 + x^2 - 2x - 3) = 2(-1)^3 + (-1)^2 - 2(-1) - 3 = -2.$$

THEOREM 2.3.7

If $\lim_{x \to c} g(x) = M$ with $M \neq 0$, then $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$,

Examples

$$\lim_{x \to 4} \frac{1}{x^2} = 16, \quad \lim_{x \to 2} \frac{1}{x^3 - 1} = \frac{1}{7}, \quad \lim_{x \to -3} \frac{1}{|x|} = \frac{1}{|-3|} = \frac{1}{3}.$$

THEOREM 2.3.8

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ with $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

Examples

$$\lim_{x \to 2} \frac{3x-5}{x^2+1} = \frac{6-5}{4+1} = \frac{1}{5}$$

$$\lim_{x \to 3} \frac{x^3-3x^2}{1-x^2} = \frac{27-27}{1-9} = 0$$

THEOREM 2.3.10

If $\lim_{x \to c} f(x) = L$ with $L \neq 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ does not exist.

Examples

From Theorem 2.3.10 you can see that

$$\lim_{x \to 1} \frac{x^2}{x-1} \qquad \lim_{x \to 2} \frac{3x-7}{x^2-4} \qquad \lim_{x \to 0} \frac{5}{x}$$

All fail to exist.

Example 1

Evaluate the limits exist:

(a) $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$, (b) $\lim\limits_{x \to 4} \dfrac{(x^2 - 3x - 4)^2}{x - 4}$, (c) $\lim\limits_{x \to -1} \dfrac{x + 1}{(2x^2 + 7x + 5)^2}$.

Example 2

Justify the following assertions.

(a) $$\lim_{x \to 2} \frac{1/x - 1/2}{x - 2} = -\frac{1}{4},$$ (b) $$\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = 6.$$

Continuity at a Point

The basic idea is as follows: We are given a function f and a number c. We calculate (if we can) both $\lim_{x\to c} f(x)$ and $f(c)$. If these two numbers are equal, we say that f is *continuous* at c. Here is the definition formally stated.

DEFINITION 2.4.1

Let f be a function defined at least on an open interval $(c-p, c+p)$. We say that *f is continuous at c* if

$$\lim_{x\to c} f(x) = f(c).$$

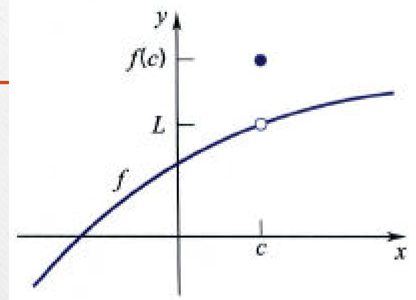

Figure 2.4.1

If the domain of f contains an interval $(c-p,\ c+p)$, then f can fail to be continuous at c for only one of two reasons: either

(i) *f has a limit as x tends to c, but* $\lim_{x\to c} f(x) \neq f(c)$, *or*

(ii) *f has no limit as x tends to c.*

In case (i) the number c is called a *removable* discontinuity. The discontinuity can be removed by redefining f at c. If the limit is L, redefine f at c to be L.

In case (ii) the number c is called an *essential* discontinuity. You can change the value of f at a billion points in any way you like. The discontinuity will remain.

The functions shown have essential discontinuities at c.

The discontinuity in Figure 2.4.2 is, for obvious reasons, called a ***jump*** discontinuity.

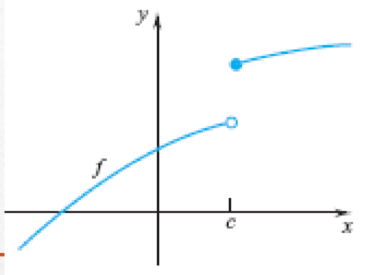

Figure 2.4.2

The functions of Figure 2.4.3 have ***infinite*** discontinuities.

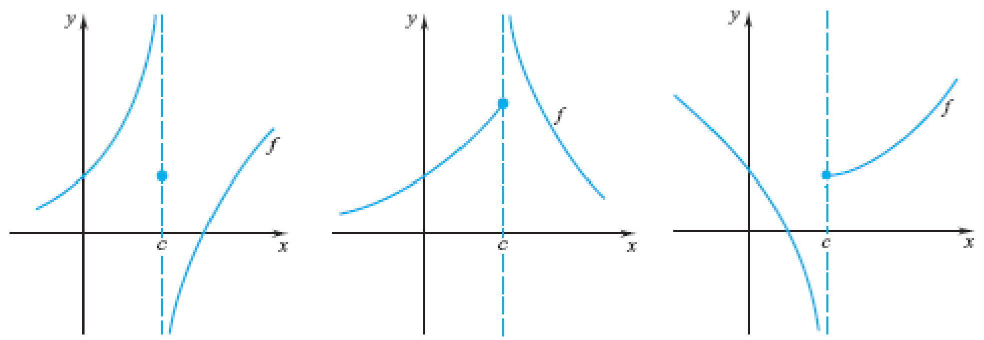

Figure 2.4.3

THEOREM 2.4.2

If f and g are continuous at c, then

(i) $f + g$ is continuous at c;

(ii) $f - g$ is continuous at c;

(iii) αf is continuous at c for each real α;

(iv) $f \cdot g$ is continuous at c;

(v) f/g is continuous at c provided $g(c) \neq 0$.

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Example 1

The function

$$F(x) = 3|x| + \frac{x^3 - x}{x^2 - 5x + 6} + 4$$

is continuous at all real numbers other than 2 and 3. You can see this by noting that

$$F = 3f + g/h + k$$

where

$$f(x) = |x|, \qquad g(x) = x^3 - x, \qquad h(x) = x^2 - 5x + 6, \qquad k(x) = 4.$$

Since f, g, h, k are everywhere continuous, F is continuous except at 2 and 3, the numbers at which h takes on the value 0. (At those numbers F is not defined.)

THEOREM 2.4.4

If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

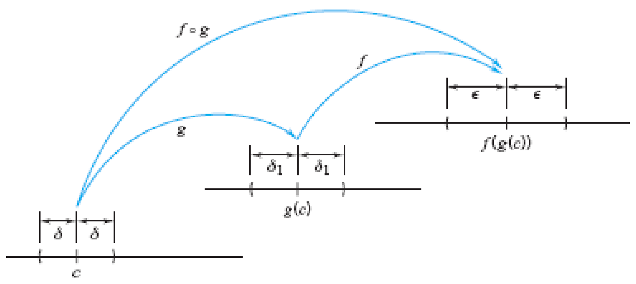

Figure 2.4.5

Example 2

The function $F(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous at all numbers greater than 3. To see this, note that $F = f \circ g$, where

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \frac{x^2+1}{x-3}.$$

Now, take any $c > 3$. Since g is a rational function and g is defined at c, g is continuous at c. Also, since $g(c)$ is positive and f is continuous at each positive number, f is continuous at $g(c)$. By Theorem 2.4.4, F is continuous at c.

Example 3

The function $F(x) = \dfrac{1}{5 - \sqrt{x^2 + 16}}$ is continuous everywhere except at x = ±3,

where it is not defined. To see this, note that $F = f \circ g \circ k \circ h$, where

$$f(x) = \frac{1}{x}, \qquad g(x) = 5 - x, \qquad k(x) = \sqrt{x}, \qquad h(x) = x^2 + 16.$$

and observe that each of these functions is being evaluated only where it is continuous. In particular, g and h are continuous everywhere, f is being evaluated only at nonzero numbers, and k is being evaluated only at positive numbers.

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DEFINITION 2.4.5 ONE-SIDED CONTINUITY

A function f is called

$$\textit{continuous from the left at } c \quad \text{if} \quad \lim_{x \to c^-} f(x) = f(c).$$

It is called

$$\textit{continuous from the right at } c \quad \text{if} \quad \lim_{x \to c^+} f(x) = f(c).$$

(2.4.6) f is continuous at c iff $f(c)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c^+} f(x)$ all exist and are equal.

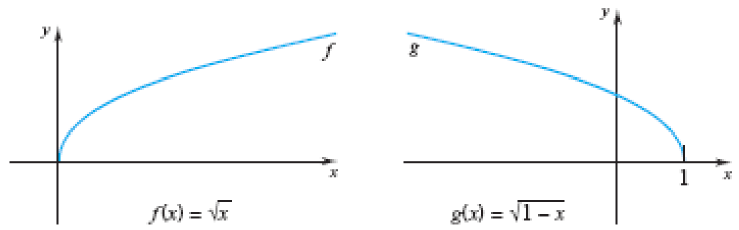

Figure 2.4.6 **Figure 2.4.7**

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Example 4

Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} 2x + 1, & x \leqq 0 \\ 1, & 0 < x \leqq 1 \\ x^2 + 1, & x > 1. \end{cases} \qquad \text{(Figure 2.4.8)}$$

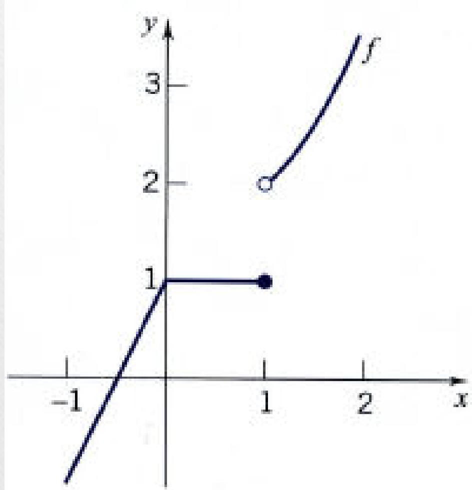

Figure 2.4.8

Example 5

Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} x^3, & x \leqq -1 \\ x^2 - 2, & -1 < x < 1 \\ 6 - x, & 1 \leqq x < 4 \\ \dfrac{6}{7 - x}, & 4 < x < 7 \\ 5x + 2, & x \geqq 7. \end{cases}$$

Continuity on Intervals

A function f is said to be *continuous on an interval* if it is continuous at each interior point of the interval and one-sidedly continuous at whatever endpoints the interval may contain.

For example:

(i) The function

$$f(x)=\sqrt{1-x^2}$$

is continuous on $[-1, 1]$ because it is continuous at each point of $(-1, 1)$, continuous from the right at -1, and continuous from the left at 1.

The graph of the function is the semicircle.

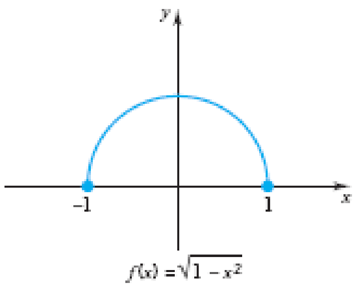

Figure 2.4.9

(ii) The function

$$f(x)=\frac{1}{\sqrt{1-x^2}}$$

is continuous on $(-1, 1)$ because it is continuous at each point of $(-1, 1)$. It is not continuous on $[-1, 1)$ because it is not continuous from the right at -1. It is not continuous on $(-1, 1]$ because it is not continuous from the left at 1.

(iii) The function graphed in Figure 2.4.8 is continuous on $(-\infty, 1]$ and continuous on $(1,\infty)$. It is not continuous on $[1,\infty)$ because it is not continuous from the right at 1.

(iv) Polynomials, being everywhere continuous, are continuous on $(-\infty,\infty)$.

Continuous functions have special properties not shared by other functions.

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THEOREM 2.5.1 THE PINCHING THEOREM

Let $p > 0$. Suppose that, for all x such that $0 < |x - c| < p$,

$$h(x) \leq f(x) \leq g(x).$$

If

$$\lim_{x \to c} h(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

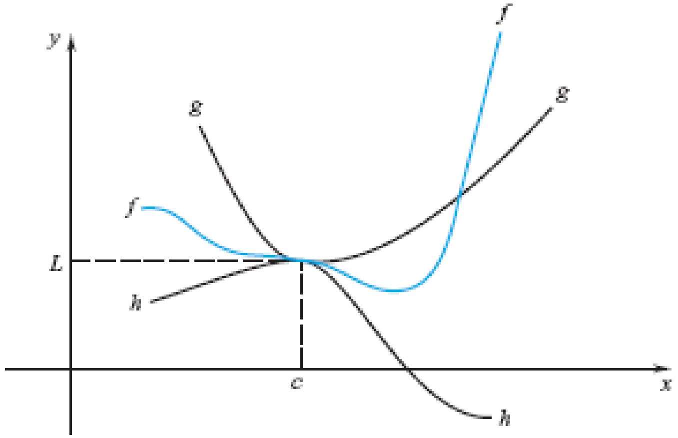

Figure 2.5.1

(2.5.2) $$\boxed{\lim_{x \to 0} \sin x = 0.}$$

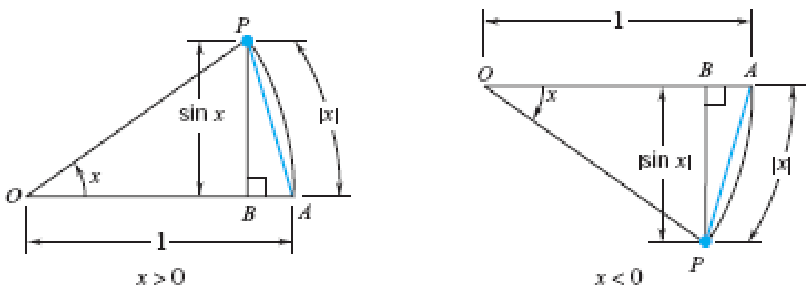

Figure 2.5.2

From this it follows readily that

(2.5.3) $$\boxed{\lim_{x \to 0} \cos x = 1.}$$

(2.5.4) $$\boxed{\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c.}$$

(2.5.5) $$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.}$$

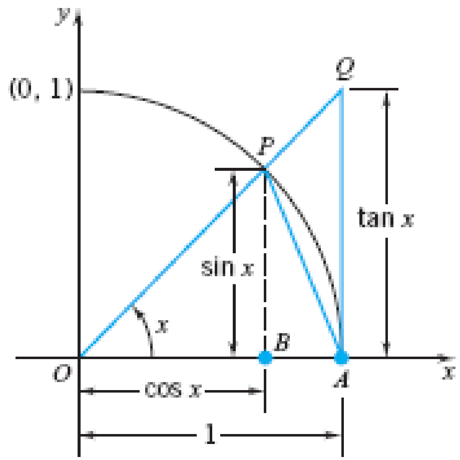

Figure 2.5.3

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In more general terms,

(2.5.6) For each number $a \neq 0$

$$\lim_{x \to 0} \frac{\sin ax}{ax} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos ax}{ax} = 0.$$

Example 1

Find

$$\lim_{x \to 0} \frac{\sin 4x}{3x} \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos 2x}{5x}$$

Solution

To calculate the first limit, we "pair off" sin 4*x* with 4*x* and use (2.5.6):

Therefore,

$$\lim_{x \to 0} \frac{\sin 4x}{3x} = \lim_{x \to 0} \left[\frac{4}{3} \cdot \frac{\sin 4x}{4x} \right] = \frac{4}{3} \lim_{x \to 0} \frac{\sin 4x}{4x} = \frac{4}{3}(1) = \frac{4}{3}$$

The second limit can be obtained the same way:

$$\lim_{x \to 0} \frac{1 - \cos 2x}{5x} = \lim_{x \to 0} \frac{2}{5} \cdot \frac{1 - \cos 2x}{2x} = \frac{2}{5} \lim_{x \to 0} \frac{1 - \cos 2x}{2x} = \frac{2}{5}(0) = 0$$

Example 2

Find $\lim_{x \to 0} x \cot 3x.$

Example 3

Find $$\lim_{x \to \pi/4} \frac{\sin(x - \frac{1}{4}\pi)}{(x - \frac{1}{4}\pi)^2}.$$

Example 4

Finding $$\lim_{x \to 0} \frac{x^2}{\sec x - 1}.$$

A function which is continuous on an interval does not “skip” any values, and thus its graph is an “unbroken curve.” There are no “holes” in it and no “jumps.” This idea is expressed coherently by the *intermediate-value theorem.*

THEOREM 2.6.1 THE INTERMEDIATE-VALUE THEOREM

If f is continuous on $[a, b]$ and K is any number between $f(a)$ and $f($
there is at least one number c in the interval (a, b) such that $f(c) =$

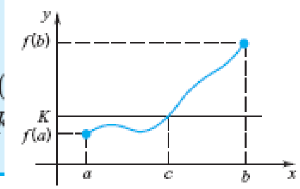
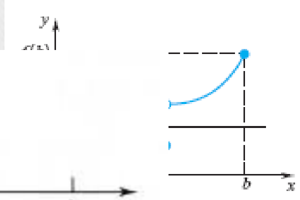


Figure 2.6.1



2.6.2

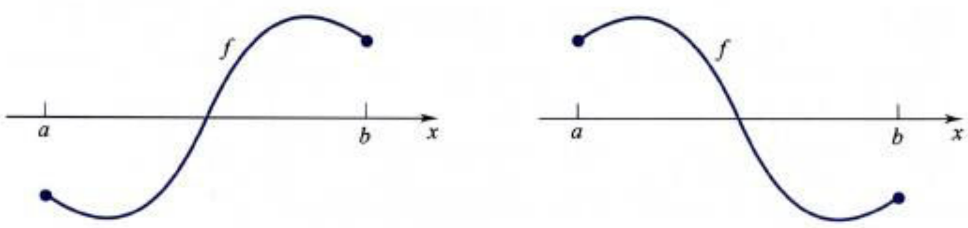


Figure 2.6.3

Example 1

We set $f(x) = x^2 - 2$. Since $f(1) = -1 < 0$ and $f(2) = 2 > 0$, there exists a number c between 1 and 2 such that $f(c) = 0$. Since f increases on [1, 2], there is only one such number. This is the number we call $\sqrt{2}$.

So far we have shown only that $\sqrt{2}$ lies between 1 and 2. We can locate $\sqrt{2}$ more precisely by evaluating f at 1.5, the midpoint of the interval [1, 2]. Since $f(1.5) = 0.25 > 0$ and $f(1) < 0$, we know that $\sqrt{2}$ lies between 1 and 1.5. We now evaluate f at 1.25, the midpoint of [1, 1.5]. Since $f(1.25) \cong -0.438 < 0$ and $f(1.5) > 0$, we know that $\sqrt{2}$ lies between 1.25 and 1.5. Our next step is to evaluate f at 1.375, the midpoint of [1.25, 1.5]. Since $f(1.375) \cong -0.109 < 0$ and $f(1.5) > 0$, we know that $\sqrt{2}$ lies between 1.375 and 1.5. We now evaluate f at 1.4375, the midpoint of [1.375, 1.5]. Since $f(1.4375) \cong 0.066 > 0$ and $f(1.375) < 0$, we know that $\sqrt{2}$ lies between 1.375 and 1.475. The average of these two numbers, rounded off two decimal places, is 1.41. A calculator gives $\sqrt{2} \cong 1.4142$. So we are not far off.

Example 2

Let

$$g(x) = \begin{cases} 1/x^2, & \\ 0, & \end{cases}$$

(Figure 2.6.5)

It is clear that g is unbounded on $[0, \infty)$. (It is unbounded above.) However, it is bounded on $[1, \infty)$. The function maps $[0, \infty)$ onto $[0, \infty)$, and it maps $[1, \infty)$ onto $(0, 1]$.

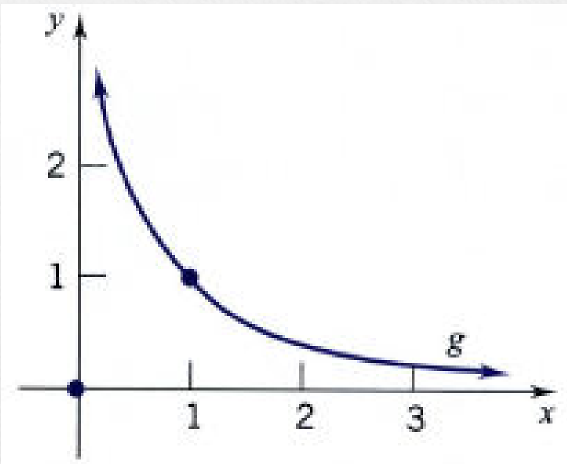


Figure 2.6.5

Main Menu

For a function continuous on a bounded closed interval, the existence of both a maximum value and a minimum value is guaranteed. The following theorem is fundamental.

THEOREM 2.6.2 THE EXTREME-VALUE THEOREM

If f is continuous on a bounded closed interval $[a, b]$, then on that interval f takes on both a maximum value M and a minimum value m.

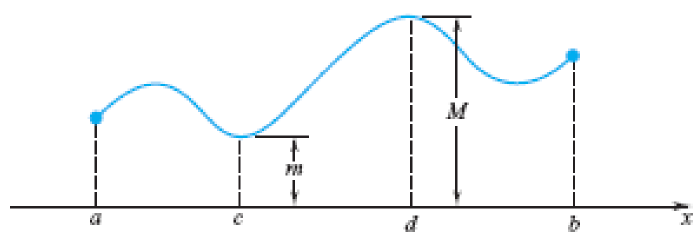


Figure 2.6.6

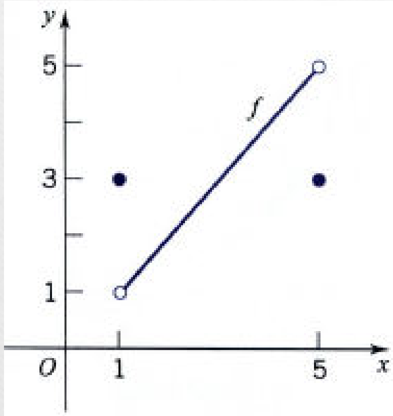


Figure 2.6.7

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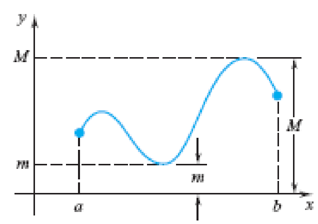
From the intermediate-value theorem we know that

"continuous functions map intervals onto intervals."

Now that we have the extreme-value theorem, we know that

"continuous functions map bounded closed intervals $[a, b]$ *onto bounded closed intervals* $[m, M]$."

Of course, if f is constant, then $M = m$ and the interval $[m, M]$ collapses to a point.



$f: [a, b] \to [m, M]$

Figure 2.6.8