Mean Value Theorem

Curriculum

- **Mean Value Theorem (MVT)**
  - Lagrange's MVT
  - Rolle's Theorem
  - Cauchy's MVT
- **Applications**

Motivation

- **Law of Mean:**

For a "smooth" curve (a curve which can be drawn in a plane without lifting the pencil on a certain interval) $y=f(x)$ $(a\le x\le b)$ it looks evident that at some point c lies between a and b i.e. $a<c<b$, the slope of the tangent $f'(c)$ will be equal to the slope of the chord joining the end points of the curve. That is, for some c lies between a and b $(a<c<b)$,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Physical Interpretation:

The velocity of a particle (matter) is exactly equal to the average speed.

Mean value Theorem (Lagrange's MVT)

- **Statement:**

Let f(x) be any real valued function defined in $a \le x \le b$ such that

(i) f(x) is continuous in $a \le x \le b$

(ii) f(x) is differentiable in $a < x < b$

Then, there exists at least one c in $a < c < b$ such that

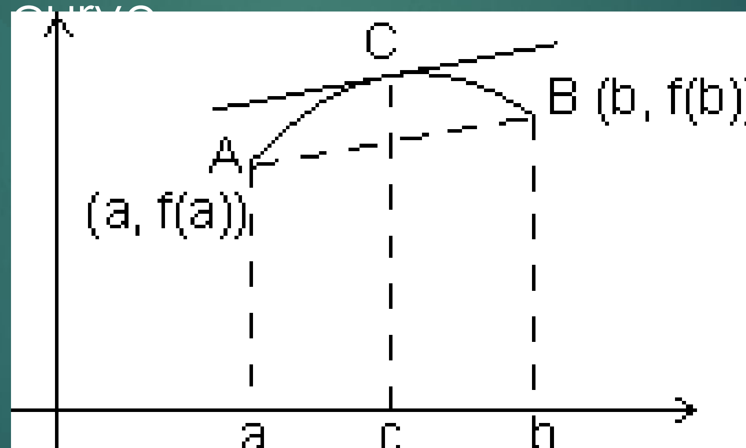
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Alternative form (by taking b=a+h, h: small increment)

$f(a+h)=f(a)+hf'(a+\theta h)$ for some θ, $0< \theta<1$

Geometrical Interpretation of MVT

- For a continuous curve $y=f(x)$ defined in $a \le x \le b$, the slope of the tangent $f'(c)$ (where c lies between a and b i.e. $a<c<b$) to the curve is parallel to the slope of the chord joining the end points of the curve.



Special Case of MVT

- **Rolle's Theorem**

Statement:

Let f(x) be any real valued function defined in $a \le x \le b$ such that

(i) f(x) is continuous in $a \le x \le b$

(ii) f(x) is differentiable in $a < x < b$

(iii) $f(a)=f(b)$

Then, there exists at least one c in $a < c < b$ such that $f'(c)=0$.

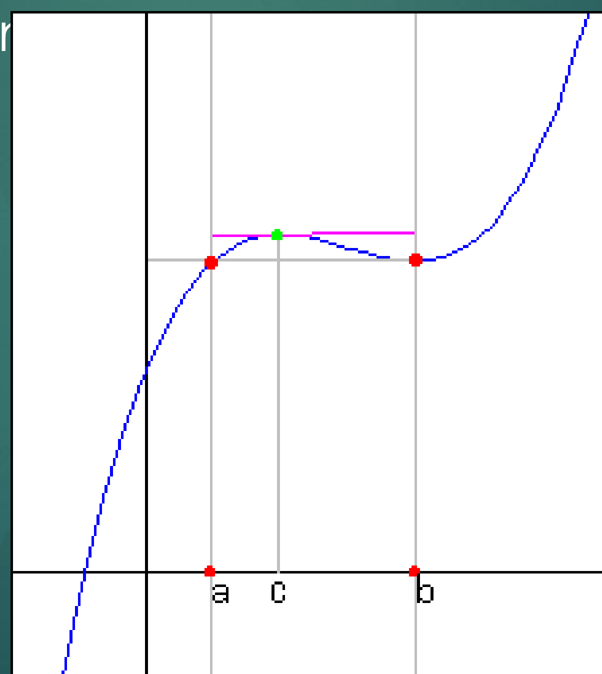
Note: All the conditions of Rolle's theorem are sufficient not necessary.

Counter Example: **i) $f(x)=2+(x-1)^{2/3}$ in $0 \le x \le 2$**

ii) $f(x)=|x-1|+|x-2|$ on [-1,3]

Geometrical Interpretation of Rolle's Theorem

- For a continuous curve y=f(x) defined in a≤x≤b, the slope of the tangent f'(c) (where c lies between a and b i.e. a<c<b) to the curve joining the two end poi … the x-axis.



General case (Cauchy's MVT)

- **Statement:**

Let f(x) and g(x) be two real valued function defined in $a \leq x \leq b$ such that

(i) f(x) and g(x) are continuous in $a \leq x \leq b$

(ii) f(x) and g(x) are both differentiable in $a < x < b$

(iii) $g'(x) \neq 0$ for some $a < x < b$

Then, there exists at least one c in $a < c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Alternative form:

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)},(0<\theta<1)$$

Interpretation (Cauchy's MVT)

- Useful generalization of the law of mean by considering a smooth curve in parametric representation x=g(t) and y=f(t) ($a \le t \le b$).
- The slope of the tangent to the curve at t=c is $\dfrac{f'(c)}{g'(c)}$
- The generalized law of mean asserts that there is always a value of c in $a<c<b$, for which the slope of the curve is equal to the slope of the tangent at c.

Observations

Rolle's Theorem
By taking f(x)=x in MVT

↓

MVT (Lagrange's)
By taking g(x)=x in Cauchy's MVT

↓

Cauchy's MVT

Applications

- To estimate some values of trignometrical function say $\sin 46^0$ etc.
- Darboux's theorem: If the interval is an open subset of R and $f: I \rightarrow R$ is differentiable at every point of I, then the range of an interval f' is an interval (not necessarily an open set).

[This has the flavour of an "Internediate Value Theorem" for f', but we are not assuming f' to be continuous].

- L' Hospital's Rule: If $f(x) \rightarrow 0$, $g(x) \rightarrow 0$ and $f'(x)/g'(x) \rightarrow L$ as $x \rightarrow c$, then $f(x)/g(x) \rightarrow L$ as $x \rightarrow c$.

- To deduce the necessary and sufficient condition of monotonic increasing or decreasing function.

For a continuous function $f:[a,b] \to R$ that is differentiable on (a,b), the following conditions are equivalent:

(i) f is increasing (or decreasing)

(ii) $f'(x) \geq 0$ (or $f'(x) \leq 0$)

- Not only the above examples but many more applications can found in different reference books from mathematics.

Books Recommended

- A first course in real analysis: Sterling K. Berberian. Springer.
- Mathematical analysis: Tom M. Apostol. Pearson Education Inc.
- An introduction to analysis: Differential calculus. Ghosh and Maity. New Central Book Agency.
- Methods of Real analysis. Richard R Goldberg. Wiley.