

Chapter 1. Newtonian Mechanics – Single Particle

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 2)

Since our course on the subject of Classical Mechanics could accurately be called “A 1001 ways of writing $\mathbf{F} = m\mathbf{a}$ ”, we will start from the beginning ...

1.1 Newton’s Laws

Newton’s Laws are usually simply stated as:

- I. *A body remains at rest or in uniform motion unless acted upon by a force.*
- II. *A body acted upon by a force moves in such a manner that the time rate of change of the momentum equals the force.*
- III. *If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.*

The First Law is meaningless without the concept of “force”, but conveys a precise meaning for the concept of “zero force”.

The Second Law is very explicit: Force is the time rate of change of the *momentum*. But what is the momentum \mathbf{p} ...

$$\mathbf{p} \equiv m\mathbf{v}, \quad (1.1)$$

with m the mass, and \mathbf{v} the velocity of the particle. We, therefore, re-write the Second Law as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}). \quad (1.2)$$

However, we still don't have a definition for the concept of "mass". This is made clear with the Third Law, which can be rewritten as follows to incorporate the appropriate definition of mass:

- III'. *If two bodies constitute an ideal, isolated system, then the accelerations of these bodies are always in opposite direction, and the ratio of the magnitudes of the accelerations is constant. This constant ratio is the inverse ratio of the masses of the bodies.*

If we have two isolated bodies, 1 and 2, then the Third Law states that

$$\mathbf{F}_1 = -\mathbf{F}_2, \quad (1.3)$$

and from the Second Law, we have

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt}, \quad (1.4)$$

or

$$m_1 \left(\frac{d\mathbf{v}_1}{dt} \right) = -m_2 \left(\frac{d\mathbf{v}_2}{dt} \right), \quad (1.5)$$

and using the acceleration \mathbf{a}

$$\begin{aligned} m_1 \mathbf{a}_1 &= -m_2 \mathbf{a}_2 \\ \frac{m_1}{m_2} &= -\frac{a_2}{a_1}. \end{aligned} \quad (1.6)$$

If one chooses m_1 as the reference or unit mass, m_2 , or the mass of any other object, can be measured by comparison (if it is allowed to interact with m_1). Incidentally, we can use equation (1.4) to provide a different interpretation of Newton's Second Law

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0 \quad (1.7)$$

or

$$\mathbf{p}_1 + \mathbf{p}_2 = \text{constant}. \quad (1.8)$$

The momentum is conserved in the interaction of two isolated particles. This is a special case of the **conservation of linear momentum**.

One should note that the Third Law is not a general law of nature. It applies when dealing with **central forces** (e.g., gravitation (in the non-relativistic limit), electrostatic, etc.), but not necessarily to other types of forces (e.g., velocity-dependent forces such as between to moving electric charges).

1.2 Frames of Reference

A reference frame is called an **inertial frame** if Newton's laws are valid in that frame. More precisely,

- If a body subject to no external forces moves in a straight line with constant velocity, or remains at rest in a reference frame, then this frame is inertial.

- If Newton's laws are valid in one reference frame, they are also valid in any other reference frame in uniform motion (or not accelerated) with respect to the first one.

The last point can be expressed mathematically like this. If the position of a free particle of mass m is represented by \mathbf{r} in a first inertial frame, and that a second frame is moving at a constant velocity $-\mathbf{V}_2$ relative to the first frame, then we can write the position \mathbf{r}' of the particle in the second frame by

$$\mathbf{r}' = \mathbf{r} + \mathbf{V}_2 t . \quad (1.9)$$

The particle's velocity \mathbf{v}' in that same frame is

$$\begin{aligned} \mathbf{v}' &= \frac{d}{dt}(\mathbf{r} + \mathbf{V}_2 t) = \frac{d\mathbf{r}}{dt} + \mathbf{V}_2 \\ &= \mathbf{v} + \mathbf{V}_2 , \end{aligned} \quad (1.10)$$

where \mathbf{v} is the velocity of the particle in the first frame. Similarly, we can calculate the particle's acceleration in the second frame (\mathbf{a}') as a function of its acceleration (\mathbf{a}) in the first one

$$\begin{aligned} \mathbf{a}' &= \frac{d\mathbf{v}'}{dt} = \frac{d}{dt}(\mathbf{v} + \mathbf{V}_2) \\ &= \frac{d\mathbf{v}}{dt} = \mathbf{a} . \end{aligned} \quad (1.11)$$

We conclude that the second reference frame is inertial since Newton's laws are still valid in it ($\mathbf{F}' = m\mathbf{a}' = m\mathbf{a}$). This result is called **Galilean invariance**, or the **principle of Newtonian relativity**.

Newton's equations do not describe the motion of bodies in non-inertial reference frame (e.g., rotating frames). That is to say, in such frames Newton's Second Law, or the equation of motion, does not have the simple form $\mathbf{F} = m\mathbf{a}$.

1.3 Conservation Theorems

We now derive three conservation theorems that are consequences of Newton's Laws of dynamics.

1.3.1 Conservation of linear momentum

This theorem was derived in section 1.1 for the case of two interacting isolated particles (see equations (1.7) and (1.8)). We now re-write it more generally from Newton's Second Law (equation (1.2)) for cases where no forces are acting on a (free) particle

$$\dot{\mathbf{p}} = 0 \quad (1.12)$$

where $\dot{\mathbf{p}}$ is the time derivative of \mathbf{p} , the **linear momentum**. Note that equation (1.12) is a vector equation, and, therefore, applies for each component of the linear momentum. In cases where a force is applied in a well-defined direction, a component of the linear momentum vector may be conserved while another is not. For example, if we consider a constant vector \mathbf{s} such that $\mathbf{F} \cdot \mathbf{s} = 0$ (the force \mathbf{F} is in a direction perpendicular to \mathbf{s}), then

$$\dot{\mathbf{p}} \cdot \mathbf{s} = \mathbf{F} \cdot \mathbf{s} = 0. \quad (1.13)$$

If we integrate with respect to time, we find

$$\mathbf{p} \cdot \mathbf{s} = \text{constant}. \quad (1.14)$$

Equation (1.14) states that *the component of linear momentum in a direction in which the forces vanishes is constant in time.*

1.3.2 Conservation of angular momentum

The **angular momentum** \mathbf{L} of a particle with respect to an origin from which its position vector \mathbf{r} is being measured is given by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}. \quad (1.15)$$

The **torque** or **moment of force** \mathbf{N} with respect to the same origin is given by

$$\mathbf{N} \equiv \mathbf{r} \times \mathbf{F}, \quad (1.16)$$

where the force \mathbf{F} is being applied at the position \mathbf{r} . Because the force is the time derivative of the linear momentum, we can write

$$\dot{\mathbf{L}} \equiv \frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}}), \quad (1.17)$$

but $\dot{\mathbf{r}} \times \mathbf{p} = 0$, since $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{p} = m\mathbf{v}$. We, therefore, find that

$$\dot{\mathbf{L}} = \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{N}. \quad (1.18)$$

It follows that the angular momentum vector \mathbf{L} will be constant in time ($\dot{\mathbf{L}} = 0$) if no torques are applied to the particle ($\mathbf{N} = 0$). That is, *the angular momentum of a particle subject to no torque is conserved.*

Note that equation (1.18) is an equation of (angular) motion that can be written in a form similar to $\mathbf{F} = m\mathbf{a}$ if we substitute \mathbf{N} for \mathbf{F} , the moment of inertia tensor $\{\mathbf{I}\}$ for m , and the angular acceleration $\boldsymbol{\alpha}$ for \mathbf{a} . So, if we introduce the angular velocity $\boldsymbol{\omega}$, which is related to \mathbf{v} through $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, we can write

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} \\ &= m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= m[r^2\boldsymbol{\omega} - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})]\end{aligned}\tag{1.19}$$

where we used the vector relation $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$. Introducing the unit tensor $\{\mathbf{1}\}$, we can write $\boldsymbol{\omega} = \{\mathbf{1}\} \cdot \boldsymbol{\omega}$ and

$$\begin{aligned}\mathbf{L} &= m[r^2\{\mathbf{1}\} - \mathbf{r}(\mathbf{r} \cdot \{\mathbf{1}\})] \cdot \boldsymbol{\omega} \\ &= \{\mathbf{I}\} \cdot \boldsymbol{\omega}\end{aligned}\tag{1.20}$$

with the inertia tensor given by

$$\{\mathbf{I}\} = m[r^2\{\mathbf{1}\} - \mathbf{r}(\mathbf{r} \cdot \{\mathbf{1}\})].\tag{1.21}$$

If we now use equation (1.18), and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, we finally get

$$\mathbf{N} = \dot{\mathbf{L}} = \{\mathbf{I}\} \cdot \boldsymbol{\alpha}.\tag{1.22}$$

1.3.3 Conservation of energy

If we consider the resultant of all forces (i.e., the total force) applied \mathbf{F} to a particle of mass m between two points “1” and “2”, we define the work done by this force on the particle by

$$W_{12} \equiv \int_1^2 \mathbf{F} \cdot d\mathbf{r}.\tag{1.23}$$

We can rewrite the integrand as

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= m \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} dt = m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt \\ &= \frac{m}{2} \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dt = \frac{m}{2} \frac{d}{dt} (v^2) dt \\ &= d\left(\frac{1}{2}mv^2\right).\end{aligned}\tag{1.24}$$

Since equation (1.24) is an exact differential, we can integrate equation (1.23) and find the work done on the particle by the total force

$$\begin{aligned} W_{12} &= \left(\frac{1}{2} m v^2 \right) \Big|_1^2 = \frac{1}{2} m (v_2^2 - v_1^2) \\ &= T_2 - T_1, \end{aligned} \quad (1.25)$$

where $T \equiv \frac{1}{2} m v^2$ is the **kinetic energy** of the particle. The particle has done work when $W_{12} < 0$.

Similarly, we can also define the **potential energy** of a particle as the work required, from the force \mathbf{F} , to transport the particle from point “1” to point “2” when there is no change in its kinetic energy. We call this type of forces *conservative* (e.g., gravity). That is

$$\int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv U_1 - U_2, \quad (1.26)$$

Where U_i is the potential energy at point “ i ”. The work done in moving the particle is simply the difference in the potential energy at the two end points. Equation (1.26) can be expressed differently if we consider \mathbf{F} as being the gradient of the scalar function U (i.e., the potential energy)

$$\mathbf{F} = -\nabla U. \quad (1.27)$$

The potential energy is, therefore, defined only within an additive constant. Furthermore, in most systems of interest the potential energy is a function of position and time, i.e., $U = U(\mathbf{r}, t)$, not the velocity $\dot{\mathbf{r}}$.

We now define the **total energy** E of a particle as the sum of its kinetic and potential energies

$$E \equiv T + U. \quad (1.28)$$

The total derivative of E is

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}. \quad (1.29)$$

Since we know from equation (1.24) that $dT = \mathbf{F} \cdot d\mathbf{r}$, then

$$\frac{dT}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \dot{\mathbf{r}}. \quad (1.30)$$

On the other hand

$$\begin{aligned}\frac{dU}{dt} &= \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial U}{\partial t} \\ &= (\nabla U) \cdot \dot{\mathbf{r}} + \frac{\partial U}{\partial t}.\end{aligned}\tag{1.31}$$

Inserting equations (1.30) and (1.31) in equation (1.29), we find

$$\begin{aligned}\frac{dE}{dt} &= \mathbf{F} \cdot \dot{\mathbf{r}} + (\nabla U) \cdot \dot{\mathbf{r}} + \frac{\partial U}{\partial t} \\ &= (\mathbf{F} + \nabla U) \cdot \dot{\mathbf{r}} + \frac{\partial U}{\partial t} \\ &= \frac{\partial U}{\partial t}.\end{aligned}\tag{1.32}$$

The last step is justified because we are assuming that \mathbf{F} is a conservative force (i.e., $\mathbf{F} = -\nabla U$).

Furthermore, for cases where U is not an explicit function of time, we have $\frac{\partial U}{\partial t} = 0$ and

$$\frac{dE}{dt} = 0.\tag{1.33}$$

We can now state the energy conservation theorem as: *the total energy of a particle in a conservative field is a constant in time.*

Finally, we group the three conservation theorems that we derived from Newton's equations:

- I. *The total linear momentum \mathbf{p} of a particle is conserved when the total force on it is zero.*
- II. *The angular momentum of a particle subject to no torque is constant.*
- III. *The total energy of a particle in a conservative field is a constant in time.*

Problems

(The numbers refer to the problems at the end of Chapter 2 in Thornton and Marion.)

2-2. A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $\mathbf{F}(\theta, \phi)$. Write the equation of motion.

Solution

Using spherical coordinates, we can write the force applied to the particle as

$$\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi. \quad (1.34)$$

But since the particle is constrained to move on the surface of a sphere, there must exist a reaction force $-F_r \mathbf{e}_r$ that acts on the particle. Therefore, the total force acting on the particle is

$$\mathbf{F}_{total} = F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi = m\ddot{\mathbf{r}}. \quad (1.35)$$

The position vector of the particle is

$$\mathbf{r} = R\mathbf{e}_r, \quad (1.36)$$

where R is the radius of the sphere and is constant. The acceleration of the particle is

$$\mathbf{a} = \ddot{\mathbf{r}} = R\ddot{\mathbf{e}}_r. \quad (1.37)$$

We must now express $\ddot{\mathbf{e}}_r$ in terms of \mathbf{e}_r , \mathbf{e}_θ , and \mathbf{e}_ϕ . Because the unit vectors in rectangular coordinates \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 , do not change with time, it is convenient to make the calculation in terms of these quantities. Using Figure 1.1 for the definition of a spherical coordinate system we get

$$\begin{aligned} \mathbf{e}_r &= \mathbf{e}_1 \sin \theta \cos \phi + \mathbf{e}_2 \sin \theta \sin \phi + \mathbf{e}_3 \cos \theta \\ \mathbf{e}_\theta &= \mathbf{e}_1 \cos \theta \cos \phi + \mathbf{e}_2 \cos \theta \sin \phi - \mathbf{e}_3 \sin \theta \\ \mathbf{e}_\phi &= -\mathbf{e}_1 \sin \phi + \mathbf{e}_2 \cos \phi. \end{aligned} \quad (1.38)$$

Then

$$\begin{aligned} \dot{\mathbf{e}}_r &= \mathbf{e}_1 (-\dot{\phi} \sin \theta \sin \phi + \dot{\theta} \cos \theta \cos \phi) \\ &\quad + \mathbf{e}_2 (\dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi) - \mathbf{e}_3 \dot{\theta} \sin \theta \\ &= \mathbf{e}_\phi \dot{\phi} \sin \theta + \mathbf{e}_\theta \dot{\theta}. \end{aligned} \quad (1.39)$$

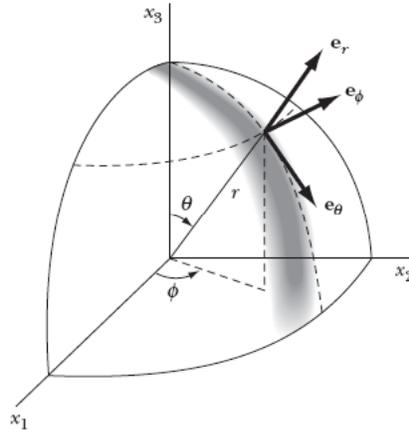


Figure 1.1 – Spherical coordinate system.

Similarly,

$$\begin{aligned}\dot{\mathbf{e}}_{\theta} &= -\mathbf{e}_r \dot{\theta} + \mathbf{e}_{\phi} \dot{\phi} \cos \theta \\ \dot{\mathbf{e}}_{\phi} &= -\mathbf{e}_r \dot{\phi} \sin \theta - \mathbf{e}_{\theta} \dot{\phi} \cos \theta.\end{aligned}\tag{1.40}$$

And, further,

$$\ddot{\mathbf{e}}_r = -\mathbf{e}_r (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \mathbf{e}_{\theta} (\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) + \mathbf{e}_{\phi} (2\dot{\theta}\dot{\phi} \cos \theta + \ddot{\phi} \sin \theta),\tag{1.41}$$

which is the only second time derivative needed. The total force acting on the particle is

$$\mathbf{F}_{total} = m\ddot{\mathbf{r}} = mR\ddot{\mathbf{e}}_r,\tag{1.42}$$

and the components are

$$\boxed{\begin{aligned}F_{\theta} &= mR(\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) \\ F_{\phi} &= mR(2\dot{\theta}\dot{\phi} \cos \theta + \ddot{\phi} \sin \theta)\end{aligned}}\tag{1.43}$$

2-32. A string connects two blocks of unequal mass over a smooth pulley. If the coefficient of friction is μ_k , what angle θ of the incline allows the masses to move at constant speed?

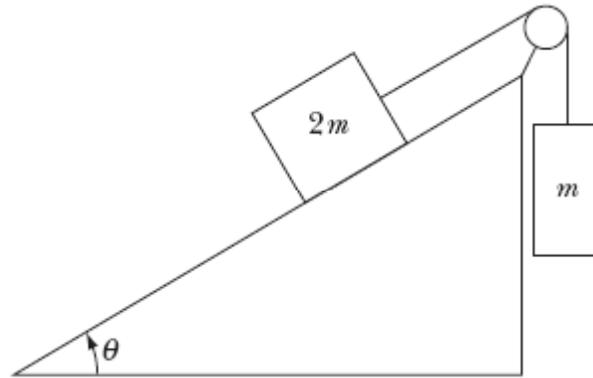


Figure 1.2 - What inclination angle will make the masses move at constant speed?

Solution

The forces on the hanging mass are easily determined from the following figure



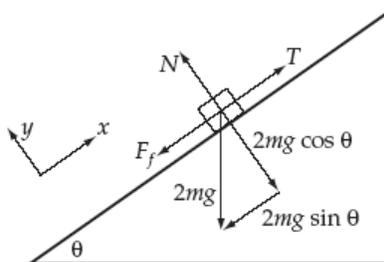
The equation of motion is (calling downward positive)

$$mg - T = ma, \tag{1.44}$$

or

$$T = m(g - a). \tag{1.45}$$

The forces on the other mass can be derived from the following figure



The y equation of motion gives

$$N - 2mg \cos \theta = m\ddot{y} = 0, \quad (1.46)$$

or

$$N = 2mg \cos \theta. \quad (1.47)$$

The x equation of motion gives (with $F_f = \mu_k N = 2\mu_k mg \cos \theta$)

$$T - 2mg \sin \theta - 2\mu_k mg \cos \theta = ma. \quad (1.48)$$

Substituting from (1.45) into (1.48)

$$mg - 2mg \sin \theta - 2\mu_k mg \cos \theta = 2ma. \quad (1.49)$$

When $\theta = \theta_0$ and $a = 0$ we get

$$g - 2g \sin \theta_0 - 2\mu_k g \cos \theta_0 = 0, \quad (1.50)$$

and

$$\begin{aligned} \frac{1}{2} &= \sin \theta_0 + \mu_k \cos \theta_0 \\ &= \sin \theta_0 + \mu_k (1 - \sin^2 \theta_0)^{1/2}. \end{aligned} \quad (1.51)$$

Isolating the square root, squaring both sides, and rearranging gives

$$(1 + \mu_k^2) \sin^2 \theta_0 - \sin \theta_0 + \left(\frac{1}{4} - \mu_k^2 \right) = 0. \quad (1.52)$$

Using the quadratic formula gives

$$\boxed{\sin \theta_0 = \frac{1 \pm \mu_k \sqrt{3 + 4\mu_k^2}}{2(1 + \mu_k^2)}} \quad (1.53)$$