Paper-IV
Chapter -Physical optics

Plane diffraction grating: Construction of grating:

Plane diffraction grating consists of a number of parallel and equidistant lines ruled on an optically plane and parallel glass plate by a fine diamond point. Each ruled line behaves as an opaque line while the transparent portion between two consecutive ruled lines behaves as a slit.

Grating element or grating constant:

If a be the width of a clear space and b be the width of a ruled line, then the distance (a+b) is called grating element or grating constant.

The two points in the consecutive clear spaces whose distance of separation is $(a + b)$, are called corresponding points.

Theory of grating:

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a plane diffraction grating consisting of N slits each of width a and with equal opaque space b between two successive slits. According to Huygens principle every point of the incident wavefront in the plane of the slits may be regarded as the origin of secondary spherical wavelets. The wavelets traveling at an angle θ with the normal are brought to focus at Q by a convex lens L.

We consider wave from a point P of the clean space at a distance x from the central point O. Let θ is the angle of diffraction. The path difference between the waves from O and P

$$\Delta = ON \pm OM = xsin\theta$$

So, phase difference

$$\delta = \frac{2\pi}{\lambda}\Delta$$

$$\delta = \frac{2\pi}{\lambda}xsin\theta$$

$$\delta = kx$$

where

$$k = \frac{2\pi}{\lambda}sin\theta$$

Let, the phase of wave from O be ωt
the phase of wave from P is $\omega t + \delta$
So, the displacement at Q due to wave from P is given by

$$y = re^{j(\omega t+\delta)} = re^{j(\omega t+kx)}$$

where r is the amplitude of the single wave. Let K be the number of waves coming from unit length of clean space. So, displacement at Q due to waves coming from dx of the clear space is

$$dy = Kdxre^{j(\omega t+kx)}$$

$$dy = Kre^{j(\omega t+kx)}dx$$

$$dy = Re^{j(\omega t+kx)}dx$$

where $Kr = R$ is the total amplitude of waves coming from unit length.

Total displacement at Q due to all waves from the whole of the slit

$$y = Re^{j\omega t}\left(\int_{-a/2}^{a/2} e^{jkx}dx + \int_{d-a/2}^{d+a/2} e^{jkx}dx\right) + ... + \int_{(N-1)d-a/2}^{(N-1)d+a/2} e^{jkx}dx\Big)$$

$$y = Re^{j\omega t}\frac{1}{jk}\left(\left(e^{jka/2} - e^{-jka/2}\right) + \left(e^{jk(d+a/2)} - e^{jk(d-a/2)}\right)\right) + ...+$$

$$\left(e^{jk((n-1)d+a/2)} - e^{jk((N-1)d-a/2)}\right)\Bigg)$$

$$y = Re^{j\omega t}\frac{2}{k}\left(\left(\frac{e^{jka/2} - e^{-jka/2}}{2j}\right) + e^{\frac{j2\pi d}{\lambda}}\left(\frac{e^{jka/2} - e^{-jka/2}}{2j}\right) + ...\right.$$

$$\left. +e^{\frac{j2\pi(N-1)d}{\lambda}}\left(\frac{e^{jka/2} - e^{-jka/2}}{2j}\right)\right)$$

$$y = Re^{j\omega t}\frac{2}{k}sin\frac{ka}{2}\left(1 + e^{\frac{j2\pi d}{\lambda}} + ... + e^{\frac{j2\pi(N-1)d}{\lambda}}\right)$$

$$y = Rae^{j\omega t}\frac{sin\frac{ka}{2}}{\frac{ka}{2}}\left(1 + e^{\frac{j2\pi d}{\lambda}} + ... + e^{\frac{j2\pi(N-1)d}{\lambda}}\right)$$

$$y = A_0\frac{sin\alpha}{\alpha}e^{j\omega t}\left(1 + e^{\frac{j2\pi d}{\lambda}} + ... + e^{\frac{j2\pi(N-1)d}{\lambda}}\right)$$

where

$$\alpha = \frac{ka}{2} = \frac{\pi a}{\lambda}sin\theta$$

We get

$$y = A_0\frac{sin\alpha}{\alpha}\left(\frac{e^{\frac{+jN2\pi d}{\lambda}} - 1}{e^{\frac{+j2\pi d}{\lambda}} - 1}\right)e^{j\omega t}$$

Where $A_0 = Ra$ is the total amplitude of waves coming from the slit of width a. Resultant amplitude after superposition at Q

$$A = A_0\frac{sin\alpha}{\alpha}\left(\frac{e^{\frac{+jN2\pi d}{\lambda}} - 1}{e^{\frac{+j2\pi d}{\lambda}} - 1}\right)$$

So, resultant intensity at Q

$$I = AA^* = A_0^2\frac{sin\alpha}{\alpha}\left(\frac{e^{\frac{+jN2\pi d}{\lambda}} - 1}{e^{\frac{+j2\pi d}{\lambda}} - 1}\right)\left(\frac{e^{\frac{-jN2\pi d}{\lambda}} - 1}{e^{\frac{-j2\pi d}{\lambda}} - 1}\right)$$

$$= I_0\frac{sin^2\alpha}{\alpha^2}\frac{sin^2 N\beta}{sin^2\beta}$$

$$= I_1 \times I_2$$

where

$$I_1 = I_0\frac{sin^2\alpha}{\alpha^2}.$$

This is intensity due to diffraction from the two slits.

$$I_2 = \frac{sin^2 N\beta}{sin^2\beta}.$$

This is intensity due to interference of waves from two slits behaving as two coherent sources.Here

$$\beta = \frac{\pi}{\lambda}(a+b)(sini \pm sin\theta)$$

Condition of minima:

Now the intensity

$$I = 0$$

If (i) $I_1 = 0$ or (ii)$I_2 = 0$.

(i) When $I_1 = 0$, we get diffraction minima. i.e.

$$4I_0\frac{sin^2\alpha}{\alpha^2} = 0$$

$$sin\alpha = 0 = sinm\pi$$

where $m =$
$m1, \pm 2, \pm 3, ...etc.$ Hence,

$$\alpha = m\pi$$

$$asin\theta = m\lambda$$

(ii) When $I_2 = 0$, we get interference minima. Hence,

$$sinN\beta = 0 = sin2m\frac{\pi}{2}$$

$$N\beta = m\pi$$

$$\frac{\pi}{\lambda}(a+b)sin\theta = \frac{m}{N}\pi$$

$$(a+b)sin\theta = \frac{m}{N}\lambda$$

Condition of principal maxima:

If the slit width a is very small and observation is confined to the neighborhood of the central pattern the variation of the factor $\frac{sin^2\alpha}{\alpha^2}$ is small and under this condition the maxima will be solely controlled by I_2. Hence

$$I_2 = maximum$$

$$\frac{sin^2 N\beta}{sin^2\beta} = maximum$$

when

$$\beta = m\pi$$

where $m = \pm 1, \pm 2, ...$

$$\frac{\pi}{\lambda}(a+b)sin\theta = m\pi$$

$$(a+b)sin\theta = m\lambda$$

This is the condition for principal maxima.

In the limit

$$Lt_{\beta\to m\pi}\frac{sinN\beta}{sin\beta} = NLt_{\beta\to m\pi}\frac{cosN\beta}{cos\beta} = N$$

Using L Hospital's rule, we get

$$I_2 = N^2$$

and

$$I = I_{pm} = I_0\frac{sin^2\alpha}{\alpha^2}\times N^2 = I_1 N^2$$

Conditions for secondary minima and maxima: For maxima or minima

$$\frac{dI_2}{d\beta} = 0$$

$$\frac{2NsinN\beta.cosN\beta}{sin^2\beta} - \frac{2sin^2N\beta.cos\beta}{sin^3\beta} = 0$$

$$2\frac{sin^2N\beta}{sin^2\beta}\Big(NcotN\beta - cot\beta\Big) = 0$$

Hence for maxima or minima (i)

$$\frac{sin^2 N\beta}{sin^2\beta} = 0$$

$$\frac{sinN\beta}{sin\beta} = 0$$

or (ii)

$$NcotN\beta - cot\beta = 0$$

(i) Secondary maxima:

When $sinN\beta = 0$, but $sin\beta \neq 0$,

$$\frac{sinN\beta}{sin\beta} = 0$$

Hence, intensity be zero. So, for minima

$$N\beta = \pm s\pi$$

$$\frac{\pi}{\lambda}(a+b)sin\theta = \pm\frac{s}{N}\pi$$

$$(a+b)sin\theta = \pm\frac{s}{N}\lambda$$

Here s has integral values.

(ii) Secondary maxima:

For the condition

$$NcotN\beta - cot\beta = 0$$

$$\frac{d^2 I_2}{d\beta^2} < 0(-Ve)$$

Here $\beta = m\pi$ gives the principal maxima. The values of $beta$ which satisfy the condition

$$NcotN\beta = cot\beta$$

give the position of secondary maxima.

Now,

$$NcotN\beta = cot\beta$$

$$N^2cot^2N\beta = cot^2\beta$$

$$N^2\frac{cos^2N\beta}{sin^2N\beta} = \frac{cos^2\beta}{sin^2\beta}$$

$$N^2\frac{cos^2N\beta}{cos^2\beta} = \frac{sin^2N\beta}{sin^2\beta}$$

$$N^2\frac{(1-sin^2N\beta)}{(1-sin^2\beta)} = \frac{N^2sin^2N\beta}{N^2sin^2\beta}$$

$$\frac{sin^2N\beta}{sin^2\beta} = N^2\frac{(1-sin^2N\beta)}{(1-sin^2\beta)} = \frac{N^2sin^2N\beta}{N^2sin^2\beta}$$

$$\frac{sin^2N\beta}{sin^2\beta} = \frac{N^2}{1+(N^2-1)sin^2\beta}$$

$$I_2 = \frac{N^2}{1+(N^2-1)sin^2\beta}$$

Hence, the intensity of secondary maxima is given by

$$I_{sm} = I_1 \times I_2$$

$$I_{sm} = I_1 \times \frac{N^2}{1+(N^2-1)sin^2\beta}$$

$$I_{sm} = I_{pm} \times \frac{1}{1+(N^2-1)sin^2\beta}$$

$$\frac{I_{sm}}{I_{pm}} = \frac{1}{1+(N^2-1)sin^2\beta}$$

This equation shows that as N increases, the intensity of secondary maxima relative to principal maxima decreases. When N is very large, the secondary maxima be very weak. For this, the secondary maxima are not observed with a grating having large N.

In Fig. curves are drawn by plotting $sin^2 N\beta$ against $N\beta$, $sin^2\beta$ against β and $\frac{sin^2 N\beta}{N^2 sin^2\beta}$ against s. It is shown from this fig. that the intensities of secondary maxima fall off aS we proceed towards the middle region between two consecutive principal maxima. These secondary maxima are unequally spaced and are not quite symmetrical.

Missing order in a diffraction grating:

A certain order interference maxima will be absent if a diffraction minima is produced at the same place (θ). We have m-th order principal maximum

$$(a+b)sin\theta = m\lambda \tag{1}$$

We have s-th order diffraction minimum

$$asin\theta = s\lambda \tag{2}$$

Dividing (1) and (2), we get

$$\frac{a+b}{a} = \frac{n}{s}$$

When $d = 2a$, or $a = b$, then

$$n = 2s$$

i.e.$2, 4, 6, ..etc$ interference fringes will be absent i.e. they will be missing in the diffraction pattern..

If $d = 3a$, then

$$n = 3s$$

i.e.$3, 6, 9, ...etc$ interference fringes will be absent i.e. they will be missing in the diffraction pattern..

If $a + b = a$, i.e.

$$b = 0$$

, The two slits join and all the orders of the interference maxima will be missing. The diffraction pattern is due to a single slit of width equal to $2a$.

Dispersive power of a grating:

The angular dispersive power $\frac{d\theta}{d\lambda}$ of a grating is defined as the rate of change of the angle of diffraction (θ) with the change in wavelength. We have for the m-the order bright band or principal maximum

$$(a+b)sin\theta = m\lambda$$

Differentiating w.r.t. λ , we get

$$(a+b)cos\theta d\theta = md\lambda$$

$$\frac{d\theta}{d\lambda} = \frac{m}{(a+b)cos\theta}$$

Thus

$$\frac{d\theta}{d\lambda} \propto m$$

$$\frac{d\theta}{d\lambda} \propto \frac{1}{(a+b)}$$

where $\frac{1}{(a+b)}$ is the number of ruling per unit length.

Also, $\frac{d\theta}{d\lambda}$ is large for large values of θ. When θ is small, then $cos\theta$ constant. Hence,

$$d\theta \propto d\lambda$$

i.e. angular separation between two spectral lines is proportional to the wavelength difference. Such a spectrum is known as normal spectrum.

bf Comparison of grating and prism spectra:

(i) A prism produces only one spectrum but a grating produces a number of spectra.

(ii) Prism spectrum is brighter than the grating spectrum.

(iii) In prism, the deviation of violet colour is more than the deviation of red colour. But in grating spectrum the deviation of violet is smaller than that of red.

(iv) The dispersive power of a grating is

$$\frac{d\theta}{d\lambda} = \frac{m}{(a+b)cos\theta}$$

The dispersive power of a prism

$$\frac{d\theta}{d\lambda} = \frac{\mu}{\mu - 1}$$

(v) Resolving power of grating is much greater than that of the prism.

(vi) Grating spectrum is nearly normal but the prism spectrum is never normal.

(vii) Dispersion in the grating spectrum does not depend on the material of the grating, but dispersion in prism spectrum depends on their material of the prism.

Determination of wavelength of light by using plane diffraction grating:

We have for the m-the order bright band or principal maximum

$$(a+b)sin\theta = m\lambda$$

$$\lambda = \frac{(a+b)sin\theta}{m}$$

$$\lambda = \frac{sin\theta}{Pm}$$

Where $P = \frac{1}{a+b}$ represents the number of rulings per unit length.

Note: If λ is known, then

$$P = \frac{sin\theta}{\lambda m}$$

Resolving power

The resolving power of an analyzing instrument is its ability to just separate two close spectral lines in their diffraction patterns.

Rayleigh criterion of resolution:

According to Rayleigh two equally bright point sources could be just resolved by any optical system if the distance between them is such that the central maximum in the diffraction pattern due to one source coincides exactly with the first minimum in the diffraction pattern due to other. This is known as Rayleigh criterion of resolution.

OR

The angular separation between the principal maxima of the two diffraction patterns should be equal to half the angular width of either principal maximum. Under this condition the resultant intensity distribution in the diffraction pattern shows a distinct dip as shown in fig. at a point half way between the two principal maxima and we are just able to identify them as separate.

According to Rayleigh the intensity at 'dip' is about $\frac{8}{\pi^2}$ times the intensity of either peak in the resultant intensity distribution. If the angular separation is smaller than this limiting value, the resultant intensity shows a single maximum as shown in fig.

Thus two close spectral lines are said to be just resolved when the angular separation between the principal maxima of two spectral lines in a given order is equal to half the angular width of either principal maximum.

Resolving power of a grating:

The resolving power of a grating measures its ability to distinguish two close spectral lines and is defined by $\frac{\lambda}{d\lambda}$. Where $d\lambda$ is the smallest wavelength difference. According to Rayleigh , two spectral lines of wavelength λ and $\lambda + d\lambda$ falls on the first minimum of the wavelength λ or vice versa.

Let parallel rays consisting of two wavelengths λ and $\lambda + d\lambda$ be incident on a grating having grating element $d = a + b$. Let the angle of diffraction θ. We have the condition of m-th oder principal maximum

$$(a + b)sin\theta = m\lambda \tag{1}$$

Differentiating , we get

$$(a + b)cos\theta d\theta = md\lambda$$

So, the angular separation $(d\theta)$ between the two principal maxima corresponding to λ and $\lambda + d\lambda$ as shown in fig.

$$d\theta = \frac{md\lambda}{(a + b)cos\theta} \tag{2}$$

For N number of slits in a grating for angle of diffraction θ, we get

$$N(a+b)sin\theta = Nm\lambda \tag{3}$$

For N number of slits in a grating for angle of diffraction $\theta + d\theta$, we get

$$N(a+b)sin(\theta + d\theta) = Nm\lambda + \lambda \tag{4}$$

Dividing (4) by (3) we get

$$\frac{sin(\theta + d\theta)}{sin\theta} = \frac{Nm\lambda + \lambda}{Nm\lambda}$$

$$\frac{sin\theta cosd\theta + cos\theta sind\theta}{sin\theta} = 1 + \frac{1}{Nm}$$

As $d\theta$ is very small, then $cosd\theta \to 1$ and $sind\theta \to d\theta$.

$$\frac{sin\theta + cos\theta d\theta}{sin\theta} = 1 + \frac{1}{Nm}$$

$$1 + cotd\theta = 1 + \frac{1}{Nm}$$

$$d\theta = \frac{1}{Nmcot\theta}$$

So, we get from (2)

$$d\theta = \frac{md\lambda}{(a+b)cos\theta} = \frac{1}{Nmcot\theta}$$

$$(a+b)sin\theta = Nm^2 d\lambda \tag{5}$$

From (1) and (5), we get

$$m\lambda = Nm^2 d\lambda$$

$$\frac{\lambda}{d\lambda} = Nm$$

This gives the resolving power. Substituting, the value of m , we get

$$\frac{\lambda}{d\lambda} = \frac{N(a+b)sin\theta}{\lambda}$$

$$\frac{\lambda}{d\lambda} = \frac{Wsin\theta}{\lambda}$$

Where $W = N(a+b)$, total width of ruled surface of the grating. When $\theta = 90^0$, then

$$\frac{\lambda}{d\lambda} = \frac{W sin 90^0}{\lambda}$$

$$\left.\frac{\lambda}{d\lambda}\right)_{max.} = \frac{W}{\lambda}$$

Resolving power of prism:

In this fig. S is a source of light.
L_1 is a collimating lens.
L_2 is the telescope lens.
I_1 is the principal maximum for λ
I_2 is the principal maximum for $\lambda + d\lambda$
θ is the angle of deviation for λ
t is the length of the prism.
We have first $(m = 1)$minimum of the image I_1

$$a sin d\theta = \lambda$$

$$a d\theta = \lambda$$

$$d\theta = \frac{\lambda}{a}$$

From fig. we get

$$\alpha + A + \alpha + \theta = \pi$$

$$\alpha = \frac{\pi}{2} - \frac{A + \theta}{2}$$

$$sin\alpha = sin\left(\frac{\pi}{2} - \frac{A+\theta}{2}\right)$$

$$\frac{a}{l} = cos\left(\frac{A+\theta}{2}\right)$$

Again,

$$t = Al$$

$$\frac{A}{2} = \frac{t}{2l}$$

$$sin\frac{A}{2} = sin\frac{t}{2l}$$

We have for prism refractive index

$$\mu = \frac{sin\frac{A+\theta}{2}}{sin\frac{A}{2}}$$

$$sin\frac{A+\theta}{2} = \mu sin\frac{A}{2}$$

$$\frac{1}{2}cos\frac{A+\theta}{2}\frac{d\theta}{d\lambda} = \frac{d\mu}{d\lambda}sin\frac{A}{2}$$

$$\frac{1}{2}\frac{a}{l}\frac{d\theta}{d\lambda} = \frac{d\mu}{d\lambda}\frac{t}{2l}$$

$$a\frac{d\theta}{d\lambda} = t\frac{d\mu}{d\lambda}$$

As $ad\theta = lambda$, so, we get the resolving power of prism

$$\frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda}$$

So, we see that the resolving power of prism

$$\frac{\lambda}{d\lambda} \propto t$$

$$\frac{\lambda}{d\lambda} \propto \frac{d\mu}{d\lambda}$$

Fabry-Perot Interferometer:

Fabry-Perot interferometer is based on the principle of multiple beam interference. For this it has high resolving power.

This interferometer consists of two glass plates ABCD and EFGH of which the surfaces AB and EF are plane and parallel and thinly silvered. The thickness d of the air film between the plates can be changed by moving one of the plates parallel to itself.

Production of fringes:

A ray PQ incident on the first plate ABCD at an angle θ. It will be broken up by repeated reflections from the surfaces AB and EF into a series of transmitted parallel light at the same angle θ. These emergent parallel rays are made to converge at a point P_1 on the screen S_2 by a lens L. We have the condition of brightness at P_1

$$2dcos\theta = m\lambda$$

This condition will be fulfilled by all points on a circle through P_1 having its centre at O_1. Thus the fringe be circular in form. When the angle of incidence θ changes, the order number m also changes. These are called fringes of equal inclination. When θ is large, m must be small. So, we get a number of concentric circular bright and dark fringes on the screen S_2, the order number of which decreases as we go outwards.

(i) Angular width:

We have the condition of brightness

$$2dcos\theta = m\lambda$$

$$-2dsin\theta d\theta = dm\lambda$$

Let $dm = 1$, then

$$-2dsin\theta d\theta = \lambda$$
$$d\theta = -\frac{\lambda}{2dsin\theta}$$

when θ is greater, $sin\theta$ is also greater. So, for outer fringes , $d\theta$ is small. Thus the angular separation $d\theta$ between two successive ringers decreases as the radius of rings increases.

(ii) Linear width:

We have the condition of brightness

$$2dcos\theta = m\lambda$$

$$m = \frac{2d}{\lambda}cos\theta$$
$$m = \frac{2d}{\lambda}\left(1 - \frac{\theta^2}{2}\right).$$

Let F be the focal length of the lens L. So, the radius of m-th order ring

$$R_m = f\theta.$$

So, we get

$$m = \frac{2d}{\lambda}\left(1 - \frac{R_m^2}{2f^2}\right).$$

Differentiating, we get

$$dm = \frac{2d}{\lambda}\frac{R_m dR_m}{f^2}$$

When $dm = -1$, then the change in radii between two successive maxima

$$dR_m = \frac{\lambda f^2}{2R_m d}$$

Hence, the rings between closer as R_m becomes higher.

Intensity distribution:

Let a is the complex amplitude of incident wave on the film at an angle θ.
r be the reflection co-efficient for waves reflected from the outer surface of the film.
r' be the reflection co-efficient for waves reflected from the inner surface of the film.
t and t' be the amplitude transmission co-efficient for waves transmitted inside and outside the film.

From Stoke's law

$$r = -r'$$

and

$$r^2 = 1 - tt'$$

Reflectivity of the film surfaces

$$R = r^2 = r'^2$$

and transmissivity

$$T = tt'$$

and assuming no absorption, we get

$$R + T = 1 - tt' + tt' = 1$$

Now, we have the complex amplitude of
first transmitted ray $A_1 = att'$
second transmitted ray $A_2 = att'r'^2e^{i\delta}$
third transmitted ray $A_3 = att'r'^4e^{2i\delta}$
etc. Here δ is the phase difference between two successive rays and is given by

$$\delta = \frac{2\pi}{\lambda}\mu 2dcos\theta$$

where μ is the refractive index of the medium. Now the resultant amplitude of the transmitted light is

$$A = A_1 + A_2 + A_3 + ...$$

$$A = att'\left(1 + r'^2 e^{i\delta} + r'^4 e^{2i\delta} + ...\right)$$

$$A = att'\frac{1}{1 - r'^2 e^{i\delta}}$$

$$A = a\frac{T}{1 - r'^2 e^{i\delta}}$$

So, the intensity of the resultant illumination is

$$I = AA^* = a\frac{T}{1 - r'^2 e^{i\delta}} \times a^* \frac{T}{1 - r'^2 e^{-i\delta}}$$

$$I = aa^* \frac{T^2}{1 + R^2 - R(e^{i\delta} + e^{-i\delta})}$$

$$I = I_0 \frac{T^2}{1 + R^2 - 2Rcos\delta}$$

$$I = I_0 \frac{T^2}{(1-R)^2 + 2R(1 - cos\delta)}$$

$$I = I_0 \frac{T^2}{(1-R)^2 + 4Rsin^2\frac{\delta}{2}}$$

$$I = I_0 \frac{T^2}{(1-R)^2} \frac{1}{1 + Fsin^2\frac{\delta}{2}}$$

Where $F = \frac{4R}{(1-R)^2}$ is known as the coefficient of fines which determines the sharpness of fringes.

Maxima:

When $\delta = 2m\pi, m = 0, 1, 2, 3, ...$ then $F = 1$. Hence

$$I_{max} = I = I_0 \frac{T^2}{(1-R)^2}$$

As there is no absorption,

$$R + T = 1$$

$$T = 1 - R$$

So, for maximum intensity

$$I_{max} = I_0 \frac{(1-R)^2}{(1-R)^2} = I_0$$

Minima:

When $\delta = (2m+1)\pi, m = 0, 1, 2, 3, ...$ then $F = \frac{4R}{(1-R)^2}$. Hence

$$I_{min} = I = I_0 \frac{T^2}{(1+R)^2}$$

As there is no absorption,

$$R + T = 1$$

$$T = 1 - R$$

So, for minimum intensity

$$I_{min} = I = I_0 \frac{(1-R)^2}{(1+R)^2}$$

Visibility of fringes:

The visibility V is defined as

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2R}{1+R^2}$$

So, in Fabry-Perot interferometer, the visibility of fringes depends only on the reflecting power.

Sharpness of fringes:

A measure of sharpness is given by the width of the fringes at points where intensity falls to half of its maximum value. i.e. when $\delta = 2m\pi \pm \delta_{1/2}$, then

$$I = \frac{I_{max}}{2}$$

We have

$$I = I_0 \frac{T^2}{(1-R)^2} \frac{1}{1+Fsin^2\frac{\delta}{2}}$$

$$I = \frac{I_{max}}{1+Fsin^2\delta_{1/2}}$$

$$\frac{I_{max}}{2} = \frac{I_{max}}{1+Fsin^2\delta_{1/2}}$$

$$\delta_{1/2} = 2sin^{-}\Big(\frac{1}{\sqrt{F}}\Big)$$

$$\delta_{1/2} = 2\Big(\frac{1}{\sqrt{F}}\Big)$$

Therefore , fringe half width

$$W = 2\delta_{1/2} = \frac{4}{\sqrt{F}}$$

To get sharp fringe, the factor F has to be increased. This can be done by increasing the reflectivity R of the film surface. For higher reflectivity, bright fringes are extremely narrow separated from each by relatively broad regions of minima between them as shown in fig.

Use or applications of Fabry-Perot interference:

(a) Comparison of wavelengths by the method of coincidence:

Let two light of close wavelengths λ_1 and $\lambda_2(< \lambda_1)$. The separation (d) between the plates of the interferometer is so adjusted that the ring systems of two wavelengths coincide. Hence, under this condition maximum visibility of the fringes is obtained. Let

$$2d_1 = m_1\lambda_1 = m_2\lambda_2$$

where d_1 is the plate separation and m_1 and m_2 are two integers.

If separation between the plates is slowly increased, the rings separate out and visibility decreases. The plate separation is further increased un till the rings coincide again and maximum visibility is obtained. At that time, if d_2 be the plate separation, then

$$2d_2 = (m_1 + p)\lambda_1 = (m_2 + p + 1)\lambda_2$$

where p is the increment in the order number of λ_1. So,

$$2(d_2 - d_1) = p\lambda_1 = (p+1)\lambda_2$$

$$\frac{1}{\lambda_1} = \frac{p}{2(d_2 - d_1)}$$

$$\frac{1}{\lambda_2} = \frac{p+1}{2(d_2 - d_1)}$$

Hence,

$$\frac{1}{\lambda_2} - \frac{1}{\lambda_1} = \frac{p+1}{2(d_2 - d_1)} - \frac{p}{2(d_2 - d_1)}$$

$$\lambda_1 - \lambda_2 = \frac{\lambda_1\lambda_2}{2(d_2 - d_1)}$$

Measurement of wavelength:

When there is a bright fringe at the centre of the field view, the separation between the plates of the interferometer is d_1. When the plate separation is increased to d_2, we get again the bright fringe at the centre.

Let N be the number of bright fringes which cross the centre of the view field, then

$$2(d_2 - d_1) = (m_1 - m_2)\lambda$$

$$2(d_2 - d_1) = N\lambda$$

$$\lambda = \frac{2(d_2 - d_1)}{N}$$

From which λ can be determined. This method is not accurate.

Study of hyperfine structure:

Fabry-Perot interferometer can be used to investigate the hyperfine structure of spectral lines. If any fringe of wavelength λ_1 is formed in the neighborhood of the centre at an angle θ_1, then

$$2dcos\theta_1 = m\lambda_1$$

For the next outer fringe for the same wavelength

$$2dcos\theta_2 = (m-1)\lambda_1$$

Let $\lambda_2 = \lambda_1 - \Delta\lambda$ By the method of coincidence in order m , we get

$$2dcos\theta_2 = m\lambda_2 = m(\lambda_1 - \Delta\lambda)$$

$$m(\lambda_1 - \Delta\lambda) = (m-1)\lambda_1$$

$$\Delta\lambda = \frac{\lambda_1}{m}$$

$$\Delta\lambda = \frac{\lambda_1}{2dcos\theta_1}$$

when θ_1 is small, then

$$\Delta\lambda = \frac{\lambda_1}{2d}$$

Cornu's spiral:

By this method any type of diffraction phenomena can be explained by dividing the half period zone into a number of sub half period strips or half-period zones.

In this fig. S is a point source of light.
XY is the incident spherical wavefront.
O is the pole of the wavefront.
Here $SO = SA = a$ and $OP = b$.
We draw a sphere of radius $OP = b$ touching the incident wavefront at O. Now the path difference

$$\Delta = SA + AP - SOP$$

$$\Delta = SA + (AB + BP) - SOP$$

$$\Delta = a + AB + b - (a + b)$$

$$\Delta = AB = MO + ON$$

Now from the property of a circle

$$MO = \frac{AM^2}{2.SO} = \frac{h^2}{2a}$$

$$ON = \frac{BN^2}{2.OP} = \frac{h^2}{2b}$$

So, we get

$$\Delta = \frac{h^2}{2a} + \frac{h^2}{2b} = \frac{h^2}{2ab}(a + b).$$

We consider that this path difference is due to m-th half period zone. So, we get

$$\Delta = m\frac{\lambda}{2}$$

Hence,

$$\frac{h^2}{2a} + \frac{h^2}{2b} = \frac{h^2}{2ab}(a+b) = m\frac{\lambda}{2}.$$

Graphical solution:

Let the first half period zone be divided into eight sub-half period zones (e.g. $Oc_1 > c_1c_2 > c_2c_3 > c_3c_4 > c_4c_5 > c_5c_6 > c_6c_7 > c_7c_8$). Here, the resultant amplitude is the vector sum of $Oc_1 > c_1c_2 > c_2c_3 > c_3c_4 > c_4c_5 > c_5c_6 > c_6c_7 > c_7c_8$) i.e. amplitude

$$A = a_1 = Oc_1 + c_1c_2 + c_2c_3 + c_3c_4 + c_4c_5 + c_5c_6 + c_6c_7 + c_7c_8$$

There is a continuous phase change from 0 to π which is due to the continuous increase in obliquity factor from O to c_8. Again the second half period zone is also divided into again eight sub-zones as $c_8b_1 > b_1b_2 > b_2b_3 > b_3b_4 > b_4b_5 > b_5b_6 > b_6b_7 > b_7b_8$. So, the resultant amplitude

$$A = Oc_8 + c_8b_8$$

$$A = a_1 + a_2$$

Similarly, if instead of eight sub-zones , we take infinitesimal width, then we get a smooth curve. Hence, the complete vibration curves for whole wavefront will be a spiral as shown in fig. which is known as Cornu's spiral.

The property of this curve is that any point on the curve, the phase lag δ is directly proportional to the square of the distance v. i.e.

$$\delta \propto v^2$$

$$\delta = \frac{\pi}{2}v^2$$

Here, the distance v is measured along the length of the curve from the point O.

We know that for the path difference of λ, the phase difference is λ. So, for the path difference Δ, the phase difference is

$$\delta = \frac{2\pi}{\lambda}\Delta$$

$$\frac{\pi}{2}v^2 = \frac{2\pi}{\lambda}\frac{h^2}{2ab}(a+b)$$

$$v = h\sqrt{\frac{2(a+b)}{ab\lambda}}.$$

Cornu's spiral can be used for any diffraction problem.

Maxima and minima in diffraction patterns by Cornu's spiral:

Let a point P on the spiral. The distance $OP = v$. A tangent at P makes an angle δ with x-axis. For a small displacement dv, the change of co-ordinates dx and dy. Here

$$dx = dv cos\delta = dv cos\left(\frac{\pi}{2}v^2\right)$$

$$dy = dv sin\delta = dv sin\left(\frac{\pi}{2}v^2\right).$$

So, the co-ordinates x and y of the Cornu's spiral are given by

$$x = \int dx = \int_0^v cos\left(\frac{\pi}{2}v^2\right)dv$$

$$y = \int dy = \int_0^v sin\left(\frac{\pi}{2}v^2\right)dv$$

These integrals are called Fresnel's integrals. These two integrals represent the horizontal and vertical components of the resultant amplitude. So, the intensity at P

$$I = K(x^2 + y^2)$$

when the whole of the wavefront is exposed, then

$$x = \int dx = \int_0^\infty cos\left(\frac{\pi}{2}v^2\right)dv = \frac{1}{2}$$

$$y = \int dy = \int_0^\infty sin\left(\frac{\pi}{2}v^2\right)dv = \frac{1}{2}$$

So, the co-ordinate of P_1 is $(1/2, 1/2)$ and that of P_2 is $(-1/2, -1/2)$.

Case I: At the origin, i.e. when $v = 0$, then

$$x = 0, y = 0$$

The spiral passes through the origin and symmetric about the origin.

Case II: At any point on the spiral, the tangent to the curve makes an angle ϕ with x-axis, such that

$$tan\phi = \frac{dy}{dx} = \frac{dv sin\left(\frac{\pi}{2}v^2\right)}{dv cos\left(\frac{\pi}{2}v^2\right)} = tan\left(\frac{\pi}{2}v^2\right)$$

$$\phi = \frac{\pi}{2}v^2$$

When $v = 0$, then $\phi = 0$. It means the curve is parallel to the x-axis at the origin. Again,

$$d\phi = \frac{2\pi}{2}vdv$$

$$\frac{dv}{d\phi} = \frac{1}{\pi v}$$

Hence, the radius of curvature of the spiral at any point

$$R = \frac{dv}{d\phi} \propto \frac{1}{v}$$

It shows that the increase of v, the radius R of the curvature decreases. Hence it takes the shape of a spiral. Finally , for $v \to \infty$, the curve ends in a point P_1 or P_2.